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► **To cite this version:**

Audrey Repetti, Mai Quyen Pham, Laurent Duval, Emilie Chouzenoux, Jean-Christophe Pesquet.  
Euclid in a Taxicab: Sparse Blind Deconvolution with Smoothed  $\ell 1 / \ell 2$  Regularization. IEEE Signal  
Processing Letters, 2015, 22 (5), pp.539-543. 10.1109/LSP.2014.2362861 . hal-01328398

**HAL Id: hal-01328398**

**<https://ifp.hal.science/hal-01328398>**

Submitted on 7 Jun 2016

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# Euclid in a Taxicab: Sparse Blind Deconvolution with Smoothed $\ell_1/\ell_2$ Regularization

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## Abstract

The  $\ell_1/\ell_2$  ratio regularization function has shown good performance for retrieving sparse signals in a number of recent works, in the context of blind deconvolution. Indeed, it benefits from a scale invariance property much desirable in the blind context. However, the  $\ell_1/\ell_2$  function raises some difficulties when solving the nonconvex and nonsmooth minimization problems resulting from the use of such a penalty term in current restoration methods. In this paper, we propose a new penalty based on a smooth approximation to the  $\ell_1/\ell_2$  function. In addition, we develop a proximal-based algorithm to solve variational problems involving this function and we derive theoretical convergence results. We demonstrate the effectiveness of our method through a comparison with a recent alternating optimization strategy dealing with the exact  $\ell_1/\ell_2$  term, on an application to seismic data blind deconvolution.

## Index Terms

Smoothed  $\ell_1/\ell_2$  regularization, norm ratio, sparsity, blind deconvolution, nonconvex optimization, preconditioned forward-backward algorithm, seismic data processing.

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Fig. 1. Unknown seismic signal  $\bar{x}$  (top), blurred/noisy observation  $y$  (bottom).

## I. INTRODUCTION

Many experimental settings are modeled as inverse problems. They resort to estimating an unknown signal  $\bar{x} \in \mathbb{R}^N$  from observations  $y \in \mathbb{R}^N$ , through the measurement process:

$$y = \bar{h} * \bar{x} + w, \quad (1)$$

an illustration of which is provided in Fig. 1. Here,  $\bar{h} \in \mathbb{R}^S$  represents an impulse response (e.g. a linear sensor response or a “blur” convolutive point spread function),  $*$  denotes a discrete-time convolution operator (with appropriate boundary processing), and  $w \in \mathbb{R}^N$  is a realization of a random variable modeling an additive noise. Standard approaches, such as Wiener filtering and its statistical extensions [1], aim at minimizing criteria based on the squared Euclidean norm ( $\ell_2^2$ ). However, the use of the sole least squares data fidelity term is prone to noise sensitivity and the addition of an  $\ell_2^2$  regularization often leads to over-smoothed estimates. The deconvolution problem becomes blind, even more ill-posed, when the blur kernel  $\bar{h}$  is unknown, and needs to be estimated as well as the target signal. Applications include communications (equalization or channel estimation) [2], nondestructive testing [3], geophysics [4]–[6], image processing [7]–[10], medical imaging and remote sensing [11]. Blind deconvolution, being an underdetermined problem, often requires additional hypotheses. A usual approach seeks estimates  $(\hat{x}, \hat{h}) \in \mathbb{R}^N \times \mathbb{R}^S$  of  $(\bar{x}, \bar{h})$  as minimizers of the sum of a data fidelity term and additional regularization terms on the signal and on the blur kernel. Such regularization functions account for a priori assumptions one imposes on original sought objects, like sparsity, and ensure the stability of the solution. Blind deconvolution is subject to scaling ambiguity, and suggests scale-invariant contrast functions [12], [13].

A decade ago, a Taxicab-Euclidean norm ratio ( $\ell_1/\ell_2$ ) arose as a sparseness measure [14]–[17], used in NMF (Nonlinear Matrix Factorization) [18]. Earlier mentions of a one-norm/two-norm ratio deconvolution appeared in geophysics [19]. It has since been used to constrain sharp images through wavelet frame coefficients [20], or for sparse recovery [21]. Such a regularization term is moreover suggested in [22] to avoid common pitfalls in blind sparse deconvolution.

Recently, [23] proposed an alternating minimization algorithm to deal with the  $\ell_1/\ell_2$  regularization function. Its originality consists of transforming the  $\ell_1/\ell_2$  nonconvex regularization term into a convex  $\ell_1$  regularization function. This is done in a reweighted fashion, by fixing the denominator  $\ell_2$  from the previous iterate. An iterative shrinkage-thresholding algorithm finally solves the remaining  $\ell_1$  regularized problem. Although the convergence of this approach has not been deeply investigated, it appears to

be quite efficient in practice. More recently, [24] proposed a scaled gradient projection algorithm for minimizing a smooth approximation of the  $\ell_1/\ell_2$  function, however limited to the case when the sparse signal to retrieve takes nonnegative values. We generalize this idea to a parametrized Smoothed One-Over-Two (SOOT) penalty for signed, real data. We present a novel efficient method based on recent results in nonconvex optimization combining an alternating minimization strategy with a forward-backward iteration [25], [26]. Moreover, we accelerate the convergence of our algorithm by using a Majorize-Minimize (MM) approach [26]–[28]. Section II introduces the minimization problem. Section III describes the proposed method and provides convergence results. The algorithm performance, compared with [23], is discussed in Section IV for seismic data blind deconvolution. Some conclusions are drawn in Section V.

## II. OPTIMIZATION MODEL

### A. Optimization tools

Our minimization strategy relies on two optimization principles. Let  $U \in \mathbb{R}^{M \times M}$  be a symmetric positive definite (SPD) matrix. Firstly, we define the  $U$ -weighted proximity operator [29, Sec. XV.4], [30] of a proper, lower semicontinuous, convex function  $\psi: \mathbb{R}^M \rightarrow ]-\infty, +\infty]$  at  $z \in \mathbb{R}^M$ , relative to the metric induced by  $U$ , and denoted by  $\text{prox}_{U,\psi}(z)$ , as the unique minimizer of  $\psi + \frac{1}{2}\|\cdot - z\|_U^2$ , where  $\|\cdot\|_U$  denotes the weighted Euclidean norm, i.e.,  $(\forall z \in \mathbb{R}^M) \|z\|_U = (z^\top U z)^{1/2}$ . When  $U$  is equal to  $I_M$ , the identity matrix of  $\mathbb{R}^{M \times M}$ , then  $\text{prox}_{I_M,\psi}$  reduces to the original definition of the proximity operator in [31]. We refer to [32]–[34] for additional details on proximity operators. Secondly, we introduce the Majoration-Minimization (MM) principle:

**Definition 1.** Let  $\zeta: \mathbb{R}^M \rightarrow \mathbb{R}$  be a differentiable function. Let  $z \in \mathbb{R}^M$ . Let us define, for every  $z' \in \mathbb{R}^M$ ,

$$q(z', z) = \zeta(z) + (z' - z)^\top \nabla \zeta(z) + \frac{1}{2} \|z' - z\|_{U(z)}^2,$$

where  $U(z) \in \mathbb{R}^{M \times M}$  is an SPD matrix. Then,  $U(z)$  satisfies the majoration condition for  $\zeta$  at  $z$  if  $q(\cdot, z)$  is a quadratic majorant of the function  $\zeta$  at  $z$ , i.e., for every  $z' \in \mathbb{R}^M$ ,  $\zeta(z') \leq q(z', z)$ .

If function  $\zeta$  has an  $L$ -Lipschitzian gradient on a convex subset  $C \subset \mathbb{R}^M$ , with  $L > 0$ , i.e., for every  $(z, z') \in C^2$ ,  $\|\nabla \zeta(z) - \nabla \zeta(z')\| \leq L\|z - z'\|$ , then, for every  $z \in C$ , a quadratic majorant of  $\zeta$  at  $z$  is trivially obtained by taking  $U(z) = LI_M$ .

### B. Proposed criterion

From now on, definitions and properties apply for every  $x = (x_n)_{1 \leq n \leq N} \in \mathbb{R}^N$  and  $h \in \mathbb{R}^S$ , unless otherwise stated. We propose to define an estimate  $(\hat{x}, \hat{h})$  of  $(\bar{x}, \bar{h})$  as a minimizer of the following penalized criterion:

$$F(x, h) = \rho(x, h) + g(x, h) + \varphi(x), \quad (2)$$

where  $\rho(x, h) = \frac{1}{2} \|h * x - y\|^2$  is the least-squares objective function,  $g$  introduces additional a priori information on the sought objects, and  $\varphi$  models the One-Over-Two norm ratio non-convex penalty function [35], defined as the quotient of  $\ell_1(x) = \sum_{n=1}^N |x_n|$  and  $\ell_2(x) = \left(\sum_{n=1}^N x_n^2\right)^{1/2}$ . The resulting regularization term is both nonconvex and nonsmooth, so that finding a minimizer of  $F$  is a challenging task.

The smooth approximations of  $\ell_1$  and  $\ell_2$ ,  $\ell_{1,\alpha}$  (sometimes called hybrid  $\ell_1$ - $\ell_2$  or hyperbolic penalty) and  $\ell_{2,\eta}$ , are defined as follows with parametric constants  $(\alpha, \eta)$ :

$$\ell_{1,\alpha}(x) = \sum_{n=1}^N \left( \sqrt{x_n^2 + \alpha^2} - \alpha \right), \quad \ell_{2,\eta}(x) = \sqrt{\sum_{n=1}^N x_n^2 + \eta^2}.$$

Note that  $\ell_1$  and  $\ell_2$  are recovered for  $\alpha = \eta = 0$ . We thus propose to replace the nonsmooth function  $\ell_1/\ell_2$  by a manageable smooth approximation. More precisely, we employ the following surrogate function:

$$\varphi(x) = \lambda \log \left( \frac{\ell_{1,\alpha}(x) + \beta}{\ell_{2,\eta}(x)} \right), \quad (3)$$

with  $(\lambda, \beta, \alpha, \eta) \in ]0, +\infty[^4$ .

The log function both makes the penalty easier to handle and, through its concavity, tends to strengthen the sparsity promoting effect of the  $\ell_1/\ell_2$  function.  $F$  corresponds to the Lagrangian function associated with the minimization of  $\rho + g$  under the constraint

$$\log \left( \frac{\ell_{1,\alpha}(x) + \beta}{\ell_{2,\eta}(x)} \right) \leq \log(\vartheta), \quad (4)$$

for some positive constant  $\vartheta$ . Owing to the monotonicity of the log function, (4) is equivalent to  $(\ell_{1,\alpha}(x) + \beta)/\ell_{2,\eta}(x) \leq \vartheta$ , which, according to (3), can be interpreted as a smooth approximation of an  $\ell_1/\ell_2$  upper bound constraint, for  $\beta$  small enough. Finally, remark that lengthy but straightforward calculations allowed us to prove that  $\varphi$  has a Lipschitzian gradient on any bounded convex subset of  $\mathbb{R}^N$ , which is a desirable property for deriving an efficient algorithm to minimize (2). In the following, we assume that  $g$  can be

split as

$$g(x, h) = g_1(x) + g_2(h), \quad (5)$$

where  $g_1$  and  $g_2$  are (non-necessarily smooth) proper, lower semicontinuous, convex functions, continuous on their domain. Moreover, we denote by

$$f(x, h) = \rho(x, h) + \varphi(x), \quad (6)$$

the smooth part of the criterion, and  $\nabla_1 f(x, h) \in \mathbb{R}^N$  (resp.  $\nabla_2 f(x, h) \in \mathbb{R}^S$ ) the partial gradient of  $f$  with respect to its first (resp. second) argument computed at  $(x, h)$ .

### III. PROPOSED ALTERNATING OPTIMIZATION METHOD

#### A. Proposed SOOT algorithm

To minimize (2), one can exploit the block-variable structure of  $F$  by using an alternating forward-backward algorithm [25], [26], [36]–[38]. At each iteration  $k \in \mathbb{N}$ , this algorithm updates  $x^k$  (resp.  $h^k$ ) with a gradient step on  $f(\cdot, h^k)$  (resp.  $f(x^{k+1}, \cdot)$ ) followed by a proximity step on  $g_1$  (resp.  $g_2$ ).

We use this alternating minimization method combined with an MM strategy, as described in [26]. For every  $(x, h) \in \mathbb{R}^N \times \mathbb{R}^S$ , let us assume the existence of SPD matrices  $A_1(x, h) \in \mathbb{R}^{N \times N}$  and  $A_2(x, h) \in \mathbb{R}^{S \times S}$  such that  $A_1(x, h)$  (resp.  $A_2(x, h)$ ) satisfies the majoration condition for  $f(\cdot, h)$  at  $x$  (resp.  $f(x, \cdot)$  at  $h$ ). Then, the SOOT algorithm for the minimization of (2) is described in Algorithm 1. Note that PALM

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#### Algorithm 1 SOOT algorithm.

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For every  $k \in \mathbb{N}$ , let  $J_k \in \mathbb{N}^*$ ,  $I_k \in \mathbb{N}^*$  and let  $(\gamma_x^{k,j})_{0 \leq j \leq J_k - 1}$  and  $(\gamma_h^{k,i})_{0 \leq i \leq I_k - 1}$  be positive sequences. Initialize with  $x^0 \in \text{dom } g_1$  and  $h^0 \in \text{dom } g_2$ .

**Iterations:**

For  $k = 0, 1, \dots$

$$\left[ \begin{array}{l} x^{k,0} = x^k, \quad h^{k,0} = h^k, \\ \text{For } j = 0, \dots, J_k - 1 \\ \quad \left[ \begin{array}{l} \tilde{x}^{k,j} = x^{k,j} - \gamma_x^{k,j} A_1(x^{k,j}, h^k)^{-1} \nabla_1 f(x^{k,j}, h^k), \\ x^{k,j+1} = \text{prox}_{(\gamma_x^{k,j})^{-1} A_1(x^{k,j}, h^k), g_1}(\tilde{x}^{k,j}), \end{array} \right. \\ x^{k+1} = x^{k, J_k}. \\ \text{For } i = 0, \dots, I_k - 1 \\ \quad \left[ \begin{array}{l} \tilde{h}^{k,i} = h^{k,i} - \gamma_h^{k,i} A_2(x^{k+1}, h^{k,i})^{-1} \nabla_2 f(x^{k+1}, h^{k,i}), \\ h^{k,i+1} = \text{prox}_{(\gamma_h^{k,i})^{-1} A_2(x^{k+1}, h^{k,i}), g_2}(\tilde{h}^{k,i}), \end{array} \right. \\ h^{k+1} = h^{k, I_k}. \end{array} \right.$$


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algorithm [25] is recovered as a special case if  $J_k \equiv I_k \equiv 1$  and, at each iteration, the Lipschitz constant

of  $\nabla_1 f(\cdot, h^k)$  (resp.  $\nabla_2 f(x^{k+1}, \cdot)$ ) is substituted for  $A_1(x^{k,0}, h^k)$  (resp.  $A_2(x^{k+1}, h^{k,0})$ ). However, recent works on variable metric strategies [26], [28] show that the use of more judicious preconditioning matrices can significantly accelerate the convergence of the algorithm. An example of such matrices is proposed in Section III-B. Moreover, we show in our experimental part the practical interest in terms of convergence speed of taking the number of inner loops  $(I_k)_{k \in \mathbb{N}}$  or  $(J_k)_{k \in \mathbb{N}}$  greater than one.

The convergence of Algorithm 1 can be derived from the general results established in [26]:

**Proposition 1.** *Let  $(x^k)_{k \in \mathbb{N}}$  and  $(h^k)_{k \in \mathbb{N}}$  be sequences generated by Algorithm 1. Assume that:*

1) *There exists  $(\underline{\nu}, \bar{\nu}) \in ]0, +\infty[^2$  such that, for all  $k \in \mathbb{N}$ ,*

$$\begin{aligned} (\forall j \in \{0, \dots, J_k - 1\}) \quad \underline{\nu} \mathbf{I}_N &\preceq A_1(x^{k,j}, h^k) \preceq \bar{\nu} \mathbf{I}_N, \\ (\forall i \in \{0, \dots, I_k - 1\}) \quad \underline{\nu} \mathbf{I}_S &\preceq A_2(x^{k+1}, h^{k,i}) \preceq \bar{\nu} \mathbf{I}_S. \end{aligned}$$

2) *Step-sizes  $(\gamma_x^{k,j})_{k \in \mathbb{N}, 0 \leq j \leq J_k - 1}$  and  $(\gamma_h^{k,i})_{k \in \mathbb{N}, 0 \leq i \leq I_k - 1}$  are chosen in the interval  $[\underline{\gamma}, 2 - \bar{\gamma}]$  where  $\underline{\gamma}$  and  $\bar{\gamma}$  are some given positive real constants.*

3)  *$g$  is a semi-algebraic function.*<sup>1</sup>

*Then, the sequence  $(x^k, h^k)_{k \in \mathbb{N}}$  converges to a critical point  $(\hat{x}, \hat{h})$  of (2). Moreover,  $(F(x^k, h^k))_{k \in \mathbb{N}}$  is a nonincreasing sequence converging to  $F(\hat{x}, \hat{h})$ .*

### B. Construction of the quadratic majorants

The numerical efficiency of the SOOT algorithm relies on the use of quadratic majorants providing tight approximations to the criterion and whose curvature matrices are simple to compute. The following proposition allows us to propose SPD matrices  $A_1$  and  $A_2$  for building majorizing approximations of  $f$  with respect to  $x$  and  $h$ .

**Proposition 2.** *For every  $(x, h) \in \mathbb{R}^N \times \mathbb{R}^S$ , let*

$$\begin{aligned} A_1(x, h) &= \left( L_1(h) + \frac{9\lambda}{8\eta^2} \right) \mathbf{I}_N + \frac{\lambda}{\ell_{1,\alpha}(x) + \beta} A_{\ell_{1,\alpha}}(x), \\ A_2(x, h) &= L_2(x) \mathbf{I}_S, \end{aligned}$$

<sup>1</sup>Semi-algebraicity is a property satisfied by a wide class of functions, which means that their graph is a finite union of sets defined by a finite number of polynomial inequalities. In particular, it is satisfied for the SOOT penalty, for standard numerical implementations of the log function.

where

$$A_{\ell_{1,\alpha}}(x) = \text{Diag} \left( \left( (x_n^2 + \alpha^2)^{-1/2} \right)_{1 \leq n \leq N} \right), \quad (7)$$

and  $L_1(h)$  (resp.  $L_2(x)$ ) is a Lipschitz constant for  $\nabla_1 \rho(\cdot, h)$  (resp.  $\nabla_2 \rho(x, \cdot)$ ).<sup>2</sup> Then,  $A_1(x, h)$  (resp.  $A_2(x, h)$ ) satisfies the majoration condition for  $f(\cdot, h)$  at  $x$  (resp.  $f(x, \cdot)$  at  $h$ ).

*Proof.* Let us decompose  $\varphi = \varphi_1 + \varphi_2$  with  $\varphi_1(x) = \lambda \log(\ell_{1,\alpha}(x) + \beta)$  and  $\varphi_2(x) = -\lambda \log(\ell_{2,\eta}(x))$ . It then suffices to prove that, for every  $x \in \mathbb{R}^N$ ,

- (i)  $A_{\varphi_1}(x) = \frac{\lambda}{\ell_{1,\alpha}(x) + \beta} A_{\ell_{1,\alpha}}(x)$  satisfies the majoration condition for  $\varphi_1$  at  $x$ ,
- (ii)  $\varphi_2$  has a  $\mu$ -Lipschitzian gradient, with  $\mu = \frac{9\lambda}{8\eta^2}$ .

On the one hand, setting  $\tau(x) = \ell_{1,\alpha}(x) + \beta$ , we have [39]

$$\tau(x') \leq \tau(x) + (x' - x)^\top \nabla \tau(x) + \frac{1}{2} \|x' - x\|_{A_{\ell_{1,\alpha}}(x)}^2, \quad (8)$$

for every  $x' \in \mathbb{R}^N$ , where  $A_{\ell_{1,\alpha}}(x)$  is given by (7).

On the other hand, for every  $(u, v) \in ]0, +\infty[^2$ ,

$$\log v \leq \log u + \frac{v}{u} - 1 = \log u + \frac{v - u}{u}. \quad (9)$$

By taking  $v = \tau(x') > 0$  and  $u = \tau(x) > 0$ , and by combining (8) and (9), we obtain

$$\varphi_1(x') \leq \varphi_1(x) + \frac{\lambda}{\tau(x)} (x' - x)^\top \nabla \tau(x) + \frac{1}{2} (x' - x)^\top \frac{\lambda}{\tau(x)} A_{\ell_{1,\alpha}}(x) (x' - x).$$

Thus, Statement (i) is proved by remarking that  $\nabla \varphi_1(x) = \frac{\lambda}{\tau(x)} \nabla \tau(x)$  and  $A_{\varphi_1}(x) = \frac{\lambda}{\tau(x)} A_{\ell_{1,\alpha}}(x)$ . On the other hand, the Hessian of  $\varphi_2$  is given by

$$\nabla^2 \varphi_2(x) = \frac{2\lambda}{\ell_{2,\eta}^4(x)} x x^\top - \frac{\lambda}{\ell_{2,\eta}^2(x)} I_N.$$

Noting that  $\ell_{2,\eta}^2(x) = \|x\|^2 + \eta^2$ , and applying the triangular inequality yield

$$\|\nabla^2 \varphi_2(x)\| \leq \frac{2\lambda \|x\|^2}{(\|x\|^2 + \eta^2)^2} + \frac{\lambda}{\|x\|^2 + \eta^2} = \chi(\|x\|),$$

where  $\chi: u \in [0, +\infty[ \mapsto \lambda \frac{3u^2 + \eta^2}{(u^2 + \eta^2)^2}$ . The derivative of  $\chi$  is given, for every  $u \in [0, +\infty[$ , by

$$\dot{\chi}(u) = \lambda \frac{2u}{(u^2 + \eta^2)^3} (\eta^2 - 3u^2),$$

<sup>2</sup>Such Lipschitz constants are straightforward to derive since  $\rho$  is a quadratic cost.

thus  $\chi$  is an increasing function on  $[0, \eta/\sqrt{3}]$  and a decreasing function on  $] \eta/\sqrt{3}, +\infty[$ , and  $\sup_{u \in [0, +\infty[} \chi(u) = \chi(\eta/\sqrt{3}) = \frac{9\lambda}{8\eta^2}$ . Hence, the proof of Statement (ii). ■

#### IV. APPLICATION TO SEISMIC DATA DECONVOLUTION

##### A. Problem statement

As some of the earliest mentions of  $\ell_1/\ell_2$  deconvolution appeared in geophysics [19], blind seismic deconvolution (or inversion [40], [41]) is a natural application. The sparse seismic signal  $\bar{x}$ , of length  $N = 784$ , on the top of Fig. 1 is composed of a sequence of spikes termed primary reflection coefficients [42]. This reflectivity series indicates, in reflection seismology at normal incidence, the travel time of seismic waves between two seismic reflectors, and the amplitude of the seismic events reflected back to the sensor. The observed seismic trace  $y$  displayed in Fig. 1-bottom follows Model (1). In this context, the blur  $\bar{h}$  is related to the generated seismic source. We use here a band-pass ‘‘Ricker’’ seismic wavelet (or Mexican hat [43]) of size  $S = 41$  (Fig. 3-bottom) with a frequency spectrum concentrated between 10 and 40 Hz. The additive noise  $w$  is a realization of a zero-mean white Gaussian noise with variance  $\sigma^2$ . Since the reflectivity series is sparse, but limited in amplitude, we choose  $g_1$  as the indicator function of the convex hypercube  $[x_{\min}, x_{\max}]^N$ . Similarly, as the seismic wavelet possesses finite energy,  $g_2$  is equal to the indicator function of the set  $\mathcal{C} = \{h \in [h_{\min}, h_{\max}]^S \mid \|h\| \leq \delta\}$ , where  $\delta > 0$ , and  $h_{\min}$  (resp.  $h_{\max}$ ) is the minimum (resp. maximum) value of  $\bar{h}$ .

##### B. Numerical results

Fig. 2 presents the variations of the reconstruction time, in seconds, with respect to the number of inner-loops  $J_k \equiv J$ , with  $I_k \equiv 1$  and noise level  $\sigma = 0.03$ . The reconstruction time corresponds to the stopping criterion  $\|x^k - x^{k-1}\| \leq \sqrt{N} \times 10^{-6}$ . One can observe that the best compromise in terms of convergence speed is obtained for an intermediate number of inner-loops, namely  $J = 71$ . Note that the quality of the reconstruction is stable for each choice of  $J$ .

Fig. 2. Reconstruction time for different numbers of inner-loops  $J_k \equiv J$  (average over thirty noise realizations).

We gather comparisons of the SOOT algorithm with [23] in Table I, where the same initialization strategy has been used for both algorithms:  $x^0$  is a constant-valued signal such that  $\|x^0\| \leq \max\{|x_{\min}|, |x_{\max}|\}$ , and  $h^0$  is a centered Gaussian filter, such that  $h^0 \in \mathcal{C}$ . Results presented in this table, for each noise level  $\sigma$ , are averaged over two hundred noise realizations. The regularization parameters

of [23] and  $(\lambda, \alpha, \beta, \eta) \in ]0, +\infty[^4$  of (3) are adjusted so as to minimize the  $\ell_1$  norm between the original and the reconstructed signals. We also set, for every  $k \in \mathbb{N}$ ,  $J_k = 71$  and  $I_k = 1$ . If both methods yield tremendous improvements in  $\ell_2$  and  $\ell_1$  norms, the SOOT algorithm exhibits better results, for all noise levels, for both  $\bar{x}$  and  $\bar{h}$  estimates, especially in terms of  $\ell_1$  norm. Interestingly, the SOOT algorithm is also significantly faster in this application.

The performance is further assessed by subjective results for  $\sigma = 0.03$ . Fig. 3-top shows the residual error of the sparse signal estimation  $\bar{x} - \hat{x}$ , for a given noise realization, where  $\hat{x}$  is estimated with [23] in (a), and with SOOT in (b). It appears, in this example, that the error is smaller using SOOT algorithm. The estimated blur kernels look similar for both methods, as displayed in Fig. 3-bottom.

Noise level ( $\sigma$ )		0.01	0.02	0.03	
Observation error	$\ell_2 (\times 10^{-2})$	7.14	7.35	7.68	
	$\ell_1 (\times 10^{-2})$	2.85	3.44	4.09	
Signal error	[23]	$\ell_2 (\times 10^{-2})$	1.23	1.66	1.84
		$\ell_1 (\times 10^{-3})$	3.79	4.69	5.30
	SOOT	$\ell_2 (\times 10^{-2})$	1.09	1.63	1.83
		$\ell_1 (\times 10^{-3})$	3.42	4.30	4.85
Kernel error	[23]	$\ell_2 (\times 10^{-2})$	1.88	2.51	3.21
		$\ell_1 (\times 10^{-2})$	1.44	1.96	2.53
	SOOT	$\ell_2 (\times 10^{-2})$	1.62	2.26	2.93
		$\ell_1 (\times 10^{-2})$	1.22	1.77	2.31
Time (s.)	[23]	106	61	56	
	SOOT	56	22	18	

TABLE I  
COMPARISON BETWEEN [23] AND SOOT FOR  $\bar{x}$  AND  $\bar{h}$  ESTIMATES (INTEL(R) XEON(R) CPU E5-2609 v2@2.5GHZ USING MATLAB 8).

Fig. 3. Top: signal estimation error  $\bar{x} - \hat{x}$  with estimates  $\hat{x}$  given by [23] (a) and SOOT (b). Bottom: Original blur  $\bar{h}$  (continuous thin blue), estimated  $\hat{h}$  with SOOT (continuous thick black) and [23] (dashed thick green).

## V. CONCLUSION

The proposed SOOT algorithm for minimizing an  $\ell_1/\ell_2$  penalized criterion has been demonstrated to be quite effective in a blind deconvolution application on seismic reflectivity data. In addition, one of its advantages is that it offers theoretically guaranteed convergence. In future works, its use should be investigated for a broader class of application areas, where norm ratios are beneficial: adaptive filtering [44], compression [45], sparse system identification [46], sparse recovery [21], or cardinality-penalized clustering [47]. The application of the method using a nonquadratic data fidelity term, in association with more sophisticated preconditioning matrices, is also of main interest.

## ACKNOWLEDGMENT

L. Duval would like to thank Igor Carron (Nuit Blanche) for useful discussions and references.

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