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Inertial flow past a finite-length axisymmetric cylinder of aspect ratio 3: effect of the yaw angle

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Abstract

We study the flow past a finite-length yawed 3D cylinder by a Finite Volume / Fictitious Domain (FV/FD) method. We validate our non-boundary-fitted method against boundary-fitted numerical results for a finite-length cylinder whose axis is parallel to the streamwise direction. Drag and lift forces exerted on the cylinder and vortex shedding onset and frequency are carefully analysed. Satisfactory agreement with published results give strong confidence in the numerical methodology provided the boundary layer is accurately resolved. Then, we carry out a detailed study of the flow past a yawed cylinder of aspect ratio $L/D = 3$ (where L is the cylinder length and D is the cylinder diameter) at moderate Reynolds numbers ($25 \leq Re \leq 250$). We show that the wake patterns and the associated transitions depend strongly on Re and the yaw angle θ with respect to the streamwise direction. Various regimes are encountered including standing-eddy pattern, steady shedding of one or two pairs of counter-rotating vortices, periodic shedding of counter-rotating vortices and unsteady shedding of hairpin shaped vortices. The steady shedding of one or two pairs of counter-rotating vortices prevails in the range of parameters studied. Hydrodynamic forces exerted on the cylinder are well approximated by laws derived in the Stokes flow regime, even for moderate Reynolds numbers. For the highest Reynolds numbers ($Re = 150, 200, 250$) we show that the forces slowly depart from the Stokes based laws. Simple modification of these laws are proposed yielding a satisfactory match with the numerical results. An accurate way to compute the torque is also proposed based on the normal force to the cylinder.

Keywords: fictitious domain method, finite-length cylinder, hydrodynamic force/torque, wake instability

1. INTRODUCTION

Fluidized beds are frequently encountered in various industrial processes such as catalysis and biomass gasification. While a large number of studies exist in the literature on the flow past a single or multiple spherical particles (see for instance Clift et al. [11] among many others), little is known about finite-length cylindrical particles that are frequently used in bubbling fluidized beds. Therefore, we investigate the flow past a single finite-length yawed cylinder using numerical simulation as a step towards the understanding of the flow around many finite-length yawed cylinders.

One of the earliest studies of the flow past a cylinder oriented perpendicularly to the streamwise direction is due to Wieselsberger [62]. Both infinite-length and finite-length cylinders were considered. The aspect ratio is defined as L/D , where L is the cylinder length and D the cylinder diameter. Wieselsberger [62] covered a large range of Reynolds numbers $Re = \rho DU/\mu$ from 400 to 8×10^5 where μ , ρ and U are the dynamic viscosity, the density and the far field inlet velocity, respectively. Wieselsberger [62] observed that the drag coefficient decreases when decreasing L/D . Zdravkovich et al. [64] studied the flow past a perpendicular cylinder of finite aspect ratio $1 \leq L/D \leq 10$ at high Reynolds numbers $6 \times 10^4 \leq Re \leq 2.6 \times 10^5$. The drag coefficient was also observed to decrease when decreasing L/D . Zdravkovich et al. [64] observed a kind of vortex shedding in the range $2 \leq L/D \leq 8$ and an asymmetric flow pattern for $1 \leq L/D \leq 3$ for all values of Re he investigated. Inoue and Sakuragi [28] performed a detailed numerical study of the flow past a finite-length cylinder. The selected aspect ratios and Reynolds numbers were respectively $0.5 \leq L/D \leq 100$ and $40 \leq Re \leq 300$. They identified five different vortex shedding patterns depending on both aspect ratio and Reynolds number. They also showed that the critical Reynolds number for the onset of the unsteady regime decreases with L/D .

The flow past an infinite yawed cylinder has also been widely studied in relation to its numerous industrial applications as, e.g., offshore risers in oil industry or strand wires of floating offshore wind turbines. Relf and Powell [48] studied experimentally the flow of air past strand wires at $Re \approx 10000$. In this regime they observed that the normal force exerted on the cylinder is proportional to the square of the normal component of the velocity (the normal direction here is given by the normal vector to the cylinder surface in the plane defined by the cylinder axis and the far field inlet velocity). Sears [53] theoretically

demonstrated, using boundary layer theory, that the flow past an infinite-length yawed cylinder is determined by the normal component of the velocity. In other words, the force on an infinite-length cylinder tilted by an angle θ with respect to the flow direction is identical to the force on the same infinite-length cylinder in cross-flow with velocity $U \sin \theta$. This law called independence principle (IP) has been widely used to predict the hydrodynamic force exerted a yawed cylinder. However this principle suffers from limitations summarized in Zdravkovich [65, p 955] and primarily due to simplifications inherent to its boundary layer derivation. Moreover its application to yawed cylinders of finite aspect ratio L/D remains an open question.

Studies of the flow past a finite-length yawed or aligned cylinder (aligned in the sense that its symmetry axis is parallel to the streamwise direction) are more sparse compared to the large amount of works on the flow past a cylinder perpendicular to the streamwise direction. Auguste [4] and Auguste et al. [5] numerically studied the wake past an aligned cylinder of small aspect ratio $0 \leq L/D \leq 1$. The Reynolds number selected was in the range $0 \leq Re \leq 400$. Auguste [4] observed that the critical Reynolds number for the transition to the unsteady regime varies strongly with the aspect ratio L/D . To the authors' knowledge, the bifurcation scenario for $L/D > 1$ has not been studied so far. Recently Chrust et al. [10] evidenced the effect of L/D on the wake past aligned spheroids. Ramberg [47] studied experimentally the flow past free-ended yawed cylinders and yawed cylinders fitted with endplates at Reynolds number $160 \leq Re \leq 1100$. He showed that results were very sensitive to the cylinder end conditions. Recently Vakil and Green [58] performed a complete analysis based on numerical results of the flow past a yawed cylinder of aspect ratio $2 \leq L/D \leq 20$ for moderate Reynolds number $1 \leq Re \leq 40$. They proposed empirical relations for the drag force and the lift force exerted on the cylinder. They also checked the validity of the IP. Even if the range of Reynolds number studied was lower than the one required for the strict application of the boundary layer theory, they obtained a relatively good agreement between the values of the drag predicted by the IP and the actual values of the drag for large $\theta > 45^\circ$.

A large number of numerical studies on the flow past a solid obstacle, regardless of its shape and angular position, use a boundary-fitted method in order to optimally compute the flow in the boundary layer around that obstacle [4, 28, 58]. These methods are very accurate but not designed to efficiently handle (i) multiple angular positions of a single solid

obstacle of arbitrary shape, (ii) a large number of solid obstacles of arbitrary shape, or (iii) a single/multiple freely moving rigid bodies. In all these 3 cases, extensive re-meshing is required, either as a function of the solid obstacle angular position [58] or as a function of the time dependent position of the particles [26]. For the 3 aforementioned flow configurations, fictitious domain methods are usually preferred. Indeed, the no-slip boundary condition on the particle surface and/or the rigid body motion inside the particle volume are imposed on a fixed Eulerian grid using forcing terms added to the fluid momentum conservation equation [35]. These methods have proven to be efficient for the flow past multiple rigid obstacles and the free motion of multiple rigid particles, with a spherical, spheroidal and angular shape (see Ardekani et al. [3], Uhlmann and Dušek [57], Wachs et al. [61] among many others).

The investigation reported here, based on about a hundred simulations spanning inclination angles θ from 0° to 90° and Reynolds numbers from 25 to 250, provides the first description of the flow past short yawed cylinders of aspect ratio 3. The selected aspect ratio and Reynolds number ranges are particularly relevant to many chemical engineering applications. In fact, cylindrical pellets with an aspect ratio around 3 are frequently encountered in fixed and fluidized bed reactors. The paper is organized as follows. The numerical procedures is presented in *Section 2*; technical details regarding the numerical schemes and the fictitious domain approach are provided in *Appendix A*. We end this section with a careful validation of our numerical method by comparing our computed results to the boundary-fitted results summarized in the Supplemental Material (SM) [SMf]. We describe and analyse the flow past a $L/D = 3$ yawed cylinder in inertial regimes in *Section 3*. A special attention is paid to the transition between wake regimes as a function of Re and θ . For the sake of conciseness, some regimes well documented in the literature as, e.g., regimes for $\theta = 90^\circ$ and $\theta = 0^\circ$), or very close to other regimes described earlier in *Section 3* are detailed in *Appendix C*. In *Section 4*, we attempt to find simple laws for the drag, lift and torque coefficients in order to provide closure laws needed in higher scale models such as Euler-Lagrange methods. In particular the validity of the IP as models derived in the Stokes regimes are discussed. Main conclusions and future work are presented in *Section 5*.

2. NUMERICAL PROCEDURES AND VALIDATION

Computations are performed with our Finite Volume / Fictitious Domain method implemented in our parallel code PeliGRIFF [60, 61]. We have used this method and corresponding code in many flow configurations, including fixed obstacles and freely-moving particles, with spherical and non-spherical particles, and with or without heat transfer. Since the whole numerical method follows almost exactly what we described in Dorai et al. [15] the numerical details are outlined in *Appendix A*. We begin this section by a presentation of the computational domain and the associated boundary conditions. The second subsection is dedicated to a detailed validation of the code in the flow configuration we are interested in here, i.e., the inertial flow past a single cylinder in an unbounded domain.

2.1. Computational domain

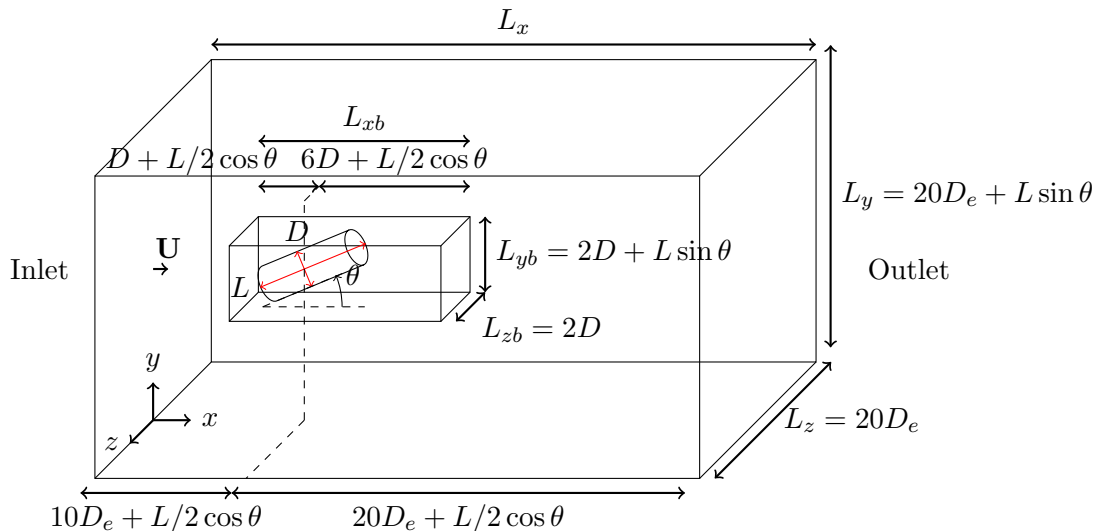


FIG. 1: Sketch of the computational domain.

The design of a computational flow domain relevant of an unbounded domain and valid in all studied configurations (various aspect ratios and yawed angles) while keeping its size "not too large" is not straightforward. In the following we briefly review several computational domains used in the past literature. Auguste [4] selected a cylindrical domain whose length and radius are $25D$ and $10D$, respectively, where D is the disk diameter. The author focused on the flow past various disks of aspect ratio varying from 0 to 1. Special attention is paid

to the distance between the disk and the outlet boundary which has to be at least $15D$ to avoid errors on the computation of the hydrodynamic force. Inoue and Sakuragi [28] studied the flow past cylinders perpendicular to the streamwise direction. In their study, the aspect ratio varied from 0.5 to 100. Their computational domain is a box and the authors defined five computational domains depending on the range of aspect ratio studied. In particular the length of the domains in the streamwise direction ranges from $115D$ to $190D$. The height of the domains, i.e., the domain size in the direction parallel to the cylinder axis, varies linearly with L as $L + 60D$. The depth of the domains, i.e., the domain size in the direction perpendicular to both the cylinder axis and the streamwise direction, is equal to $60D$ and is thus fixed for all aspect ratios studied. Vakil and Green [58] studied the flow past a yawed cylinder of variable aspect ratio ranging from 2 to 20. Their computational domain shares some similarities with the computational domain selected by Inoue and Sakuragi [28]. Indeed both the domain length and the domain height depend on L and are set to $25L$ and $12L$, respectively, while the domain depth is fixed and set to $50D$.

In this work, we define the size of our domain using a length proportional to the equivalent spherical diameter (the diameter of a sphere with same volume as the cylinder): $D_e = (LD^2)^{1/3}$. This choice ensures that the domain size changes with the size of the cylinder while remaining relatively small. This design rule remains valid from $L/D = 1$ and at least up to $L/D = 10$ as shown later in this section with careful validations.

Simulations are performed in a cuboid domain (i.e., a box) meshed with an irregular cartesian grid. (x, y, z) coordinates are defined as follows: x is parallel to the streamwise direction, y is perpendicular to the streamwise direction in the plane defined by the streamwise direction and cylinder axis, z is perpendicular to the plane defined by the streamwise direction and cylinder axis. The box size is adapted to the cylinder diameter D , to the cylinder length L and the angular position of the cylinder defined by the angle θ between the cylinder axis and the streamwise direction. Indeed the length L_x , height L_y and depth L_z of the box are respectively $30D_e + L \cos \theta$, $20D_e + L/2 \cos \theta$ and $20D_e$ (see FIG. 1). L_y and L_z are chosen sufficiently large to avoid wall effects for moderate Reynolds numbers. L_z is defined in such a way that the wake can develop without being perturbed by the outlet boundary. The domain is divided into two main regions. An inner region around the cylinder which is made of regular cells. The size of this subdomain (L_{xb}, L_{yb}, L_{zb}) is specified in FIG. 1. L_{xb} is larger downstream of the cylinder to ensure that the near wake is well

captured. The outer region is made of stretched cells of growing size away from the cylinder that smoothly match the inner region cell size.

Boundary conditions are prescribed as follows. Symmetry boundary conditions are imposed on the 4 lateral walls. At the inlet a uniform velocity profile is imposed as $\mathbf{u} = (U, 0, 0)$. The imposition of the outlet boundary condition is not straightforward and different choices can be found in the literature [45, p. 36]. The choice made in the PeliGRIFF code is to impose a zero velocity gradient condition $\partial\mathbf{u}/\partial n = 0$ and an arbitrary reference pressure $p = p_{ref}$ at the outlet boundary. Such a boundary condition has been used with success by other authors as, e.g., Kim and Elghobashi [33] to study the unsteady flow past a sphere.

In the following sections, the numerical method described above is applied to the study of the flow past a finite-length yawed cylinder. The Reynolds number $Re = \rho UD/\mu$, the yaw angle θ and the aspect ratio L/D fully characterize the system. Governing equations and flow variables are non-dimensionalized by introducing the following scales: D for length, U for velocity, D/U for time (implying that we employ a convective time scale), ρU^2 for pressure, $\rho U^2/D$ for Lagrange multiplier and U/D for vorticity. The choice of the length scale is far from straightforward. Indeed assorted definitions are used in the literature : Sears [53] used the cylinder length, Vakil and Green [58] used the cylinder diameter and Hölzer and Sommerfeld [25] used the equivalent diameter. The analysis to come involves the following dimensionless output parameters:

- The Strouhal number $St = fD/U$ when the wake and the force experienced by the body are unsteady and periodic. St compares the vortex shedding frequency f to the flow characteristic frequency U/D . Different ways of measuring f are found in the literature : local measurement of the oscillation of variables in the wake [28] or measurement of the oscillation of the lift force [4]. We use the latter definition. When there is no ambiguity on the orientation of the lift force (i.e., when the wake exhibits a symmetry plane), we write the dimensionless frequency as St . Otherwise, when the wake is fully three-dimensional and/or oscillates with multiple frequencies, we specify the direction and/or the frequency as a subscript of St [Note1]. In both cases a Discrete Fourier Transform (DFT) analysis is performed (details about the DFT analysis can be found in *Appendix B*). This procedure allows us to compute the frequencies appearing in the wake with a good accuracy . As detailed in *Appendix B* the magnitude of the error associated with the estimation of the dimensionless frequencies

is inversely proportional to the physical simulation time. In the remainder of the paper this dimensionless error is always estimated as $\pm 1/2 \times D/U \times 1/(n\Delta t)$ where n is the number of simulated time steps used in the DFT.

- To characterize the hydrodynamic force and torque exerted on the cylinder, we define the classical drag coefficient as $C_D = F_x/(1/2\rho U^2 LD)$ and lift coefficients as $C_{Li} = F_i/(1/2\rho U^2 LD)$, $i = (y, z)$. Those coefficients represent ratios of hydrodynamic force components to an inertial characteristic force $1/2\rho U^2 LD$. LD is proportional to the lateral surface area of the cylinder. It is common practice to define the drag coefficient of a bluff body using the surface area of the projection of the body on a plane normal to the streamwise direction [7, p 339]. Our choice to use LD as the reference surface area for drag and lift coefficients is guided by two main reasons: (i) when $L \gg D$ and $\theta > 0$ the projected surface area of the disk becomes negligible compared to the lateral surface area of the cylinder, and (ii) since the reference surface area does not depend on θ , comparison between hydrodynamic force and torque at different yaw angles is made simpler. Accordingly, the torque coefficient C_T is defined as $C_T = T/(1/2\rho U^2 L^2 D)$. It is also worth noting that other authors choose D_e as a single reference length for all dimensionless quantities [52].

- The wake is visualized using the Q criterion [27] defined as follows:

$$Q = \frac{1}{2} (e_{ij}e_{ij} - \omega_{ij}\omega_{ij}) \quad (1)$$

where $e_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ and $\omega_{ij} = 1/2(\partial u_i/\partial x_j - \partial u_j/\partial x_i)$ are the components of the strain-rate and vorticity tensors, respectively.

In all computations the dimensionless time step is set to $\Delta t = 1.25 \times 10^{-2}$ in order to (i) satisfy the CFL condition and (ii) make the splitting error tolerable. Each computation is run over a dimensionless time interval at least equal to $t = 100$ when the flow is steady. When the flow is unsteady and periodic, computations are run at least until $t = 200$ and until the system has experienced at least ten periods of oscillations. Time-averaged quantities are denoted with an overline symbol, i.e., time-averaged value of x is denoted \bar{x} .

In order to evaluate the influence of the domain size on the numerical results, the flow past a ($L/D = 10$, $\theta = 90^\circ$) cylinder at $Re = 100$ is computed with 2 different domain sizes: (i) the domain as defined in FIG. 1 and (ii) a domain 1.5 larger in all directions. We keep

Domain	C_D	St
Medium	3.753	0.129 ± 0.003
Large	3.789	0.130 ± 0.003
Inoue and Sakuragi [28]	-	0.124

TABLE I: Drag coefficient $C_D = F_x/(1/2\rho U^2 LD)$ and Strouhal number $St = fD/U$ for two domain sizes. The medium domain is the one shown in FIG. 1. The large domain is 1.5 larger in all directions. Data of Inoue and Sakuragi [28] are taken as reference.

the same grid size in the inner region around the cylinder for the 2 domains. The selected grid size corresponds to 32 grid points over the cylinder diameter. This space resolution is enough at $Re = 100$ to capture well the viscous boundary layer that develops along the cylinder surface. Results obtained with the two domains are compared to each other and to those of Inoue and Sakuragi [28]. TAB. I shows that the difference between our 2 domains is less than 1% for both the drag coefficient C_D and the Strouhal number St . Our computed St matches very well St computed by Inoue and Sakuragi [28]. The relative error of about 3.5% is deemed to be small given the differences between our computational method and Inoue and Sakuragi [28]’s computational method (in particular non boundary fitted versus boundary fitted, respectively). The domain size as defined in FIG. 1 is thus used for all computations presented in this paper.

2.2. Flow past a $L/D = 1$ cylinder with $\theta = 0$: comparison with boundary fitted mesh results

In this section we compare our results to results presented in the SM [SMf] obtained with a boundary-fitted method implemented in the JADIM code from IMFT that enables one to adopt an arbitrarily fine body-fitted mesh in the boundary layer around the cylinder. The JADIM numerical solution is hence deemed to be a reference solution. A cylinder of aspect ratio $L/D = 1$ aligned with the streamwise direction is considered. To demonstrate the ability of our approach to properly describe the flow past a yawed cylinder, we select 5 Reynolds numbers $Re = 25, 50, 100, 200, 360$ covering both the stationary regime with axial symmetry and the unsteady vortex shedding regime.

		C_D	$E(C_D)\%$	l_r	$E(l_r)\%$
$Re = 25$	Reference	9.2868	-	0.430	
	16 cells/ D	9.4441	+1.6933	0.453	+5.35
	32 cells/ D	9.3591	+0.77811	0.447	+3.84
$Re = 50$	Reference	6.1591	-	0.720	-
	16 cells/ D	6.2668	+1.7486	0.741	+2.99
	32 cells/ D	6.2034	+0.72013	0.732	+1.67
$Re = 100$	Reference	4.2210	-	1.12	-
	16 cells/ D	4.3533	+3.1338	1.17	+4.02
	32 cells/ D	4.2532	+0.76104	1.14	+1.43
$Re = 200$	Reference	2.9468	-	1.630	-
	16 cells/ D	3.2466	+10.173	1.86	+14.1
	32 cells/ D	3.0324	+2.9033	1.68	+2.79

TABLE II: Comparison of the drag coefficient C_D and length of standing eddy l_r given by our numerical method and the boundary-fitted method used in the SM [SMf] for $25 \leq Re \leq 200$. The number of cells distributed along the cylinder diameter varies from 16 to 32. $E(C_D)$ and $E(l_r)$ represent the relative error on the drag coefficient and the relative error on the length of the standing eddy, respectively.

TAB. II shows C_D and the recirculation length l_r for $Re = 25, 50, 100, 200$ (stationary regime) as a function of grid size. l_r is measured from the downstream tip of the cylinder to the downstream tip of the eddy. The downstream tip of the eddy is fitted with a fourth-order polynomial. The error on C_D using the coarsest grid (16 grid points per cylinder diameter) is less than 3.5% except for $Re = 200$. Since the thickness of the boundary layer scales as $\mathcal{O}(D/Re^{1/2})$, there is approximately one grid point in the viscous boundary layer at $Re = 200$. This is far from being sufficient, since even for boundary fitted mesh about 3 to 5 five grid points are necessary to accurately describe the viscous boundary layer [4]. The error on l_r using the coarsest grid is larger than 3% for all Reynolds numbers. This error decreases significantly using a twice finer grid (32 grid points per cylinder diameter). However, we note that the error made on l_r is still higher than 3% for $Re = 25$. The increase of numerical errors for low Reynolds number flows is pointed out by Wachs et al. [61] [Note2]

and is often associated to the operator splitting error that scales as $\Delta t/Re$ [42].

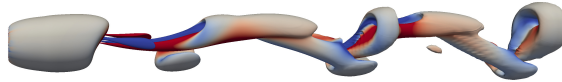


FIG. 2: Wake pattern past a $L/D = 1$ cylinder aligned with the streamwise direction at $Re = 360$. 96 cells are distributed along the cylinder diameter. The wake is visualized using the Q criterion. Isosurfaces of $Q = 10^{-3}$ are shown and are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 .

FIG. 2 shows the vortex shedding behind the cylinder at $Re = 360$. The wake exhibits a planar symmetry in the (x, y) cut plane containing the cylinder axis. Hairpin vortices are shed periodically behind the cylinder. This type of wake is a distinctive feature of wake instability and has been observed in the literature for the flow past a sphere [51], the flow past a cylinder [28] and even when a sphere crosses a fluid-fluid interface [43]. The vortex structure is single-sided, i.e., hairpin vortices always rotate in the same direction. Moreover, vortices are all shed with the same orientation which induces an averaged non-zero lift force on the cylinder (the direction of the lift force is defined unambiguously in that case owing to the wake symmetry plane).

	$\overline{C_D}$	$E(\overline{C_D})\%$	St
$Re = 360$ Reference	0.578	-	0.118
16 cells/ D	0.808	+39.8	-
32 cells/ D	0.678	+17.3	0.120 ± 0.003
64 cells/ D	0.609	+5.44	0.115 ± 0.003
96 cells/ D	0.597	+3.36	0.118 ± 0.002

TABLE III: Comparison of the drag coefficient C_D and Strouhal number St given by our numerical method and the boundary-fitted method used the SM [SMf] for $Re = 360$. The number of cells distributed along the cylinder diameter varies from 16 to 96. $E(C_D)$ represents the relative error on the drag coefficient.

TAB. III shows C_D and St for $Re = 360$ as a function of grid size. For the coarsest grid (16 grid points per cylinder diameter) the error on C_D is close to 40%. Moreover, vortices

are shed randomly and prevent from defining a characteristic vortex shedding frequency, and hence St . The error on C_D is less than 20% for a grid size corresponding to 32 grid points per cylinder diameter. The wake (not shown here) consists of hairpin vortices which are not shed periodically. Indeed a second frequency manifests in the wake (close to one fourth of the expected frequency) and is a pure numerical artefact. It remains possible to define St based on the highest frequency, the resulting error is less than 5%. The spurious frequency disappears when using a grid size corresponding to 64 grid points per cylinder diameter. For that case, TAB. III shows that the error on C_D is about 5% while the error on St is of the order of a few per cents. For the finest grid corresponding to 96 grid points per cylinder diameter, the error on C_D is less than 3.5%.

To summarize the findings of our mesh sensitivity analysis on cases representative of our physical analysis in the rest of this paper, it is necessary to distribute at least 3-4 grid points in the boundary layer around the cylinder (estimated as $D/Re^{1/2}$) to accurately describe the flow regimes we are interested in. When this condition is satisfied, our fictitious domain method yields accurate results for the flow past a yawed cylinder. Thereby, for the range of $Re \in [25, 250]$ we intend to investigate in this paper, we use either 32 grid points per cylinder diameter for the low $25 \leq Re \leq 100$ cases, 48 grid points per cylinder diameter for the moderate $100 < Re < 150$ cases, and 64 grid points per cylinder diameter for the high $Re \geq 150$ cases. Corresponding meshes comprise between 10 to 100 millions of grid cells.

3. FLOW PAST A YAWED $L/D = 3$ CYLINDER

We now investigate the flow past a $L/D = 3$ cylinder for Reynolds numbers $Re \in [25, 250]$ and yaw angles $\theta \in [0^\circ, 90^\circ]$. In total, more than eighty well resolved simulations are performed.

FIG. 3 illustrates the hydrodynamic regimes observed in a (Re, θ) plane. Colour areas delineate the regimes based on symmetry and unsteadiness. Three different temporal regimes are encountered: steady, periodic and unsteady. The regime is called periodic when the wake unsteadiness is predominantly governed by a unique frequency, otherwise it is called unsteady. Three types of symmetry can also be defined: (i) x axial symmetry, (ii) (x, z) horizontal reflectional planar symmetry, and (iii) (x, y) vertical reflectional planar symmetry. We observe seven regimes classified by unsteadiness and symmetry properties in FIG. 3. For

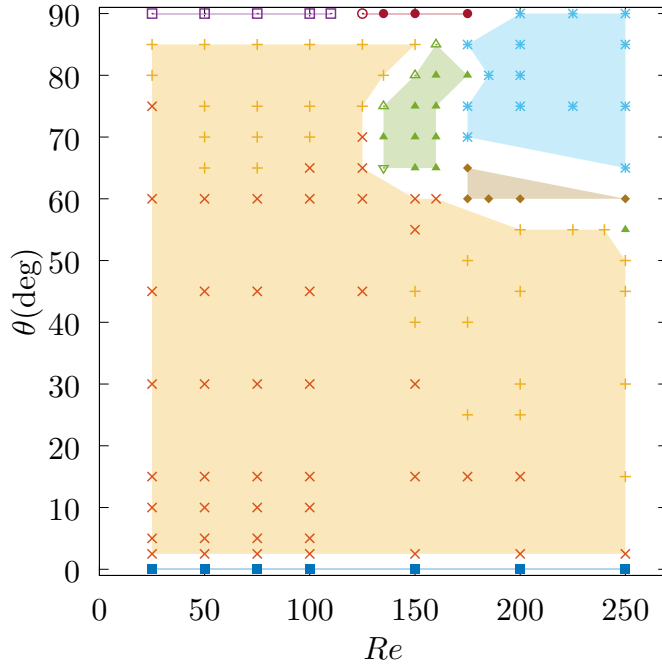


FIG. 3: Wake patterns past a $L/D = 3$ cylinder as function of θ and Re . Coloured areas represent regimes by symmetry and unsteadiness. Blue : steady axisymmetric wake, yellow : steady symmetric wake with respect to the (x, y) plane, purple : steady wake with two symmetry planes ((x, y) and (x, z)), green : periodic wake with one symmetry plane (x, y) , red : periodic wake with one symmetry plane (x, z) , brown : periodic shedding of quasi-symmetric double-sided hairpin vortices, cyan : 3D unsteady wakes. ■ : steady standing eddy or toroidal vortex, × steady shedding of one counter-rotating vortex pair, + steady shedding of 2 counter-rotating vortex pairs, □ steady shedding of 2 symmetric counter-rotating vortex pairs, △ periodic shedding of 2 counter-rotating vortex pairs ((x, y) symmetry plane), ▽ periodic shedding of one counter-rotating vortex pair ((x, y) symmetry plane), ⊙ periodic shedding of 2 counter-rotating vortex pairs ((x, z) symmetry plane), ▲ periodic shedding of single-sided hairpin shaped vortices, ● periodic shedding of symmetric double-sided hairpin shaped vortices ((x, z) symmetry plane), * unsteady shedding of asymmetric double-sided hairpin shaped vortices, ◆ periodic shedding of quasi-symmetric double-sided hairpin vortices.

$\theta = 0^\circ$, due to the specific geometric configuration, the wake is steady and axisymmetric for all Re studied. It is likely that the axisymmetry will be broken for sufficiently higher $Re > 250$ as observed for the $L/D = 1$ cylinder (see SM [SMf]). For $\theta > 0^\circ$ the axisymmetry

is intrinsically broken but the wake keeps a steady vertical symmetry plane. This regime (in yellow in FIG. 3) is the most frequently observed regime for the range of Re studied. We also would like to shortly underscore the specific symmetry observed when the cylinder is perpendicular to the flow direction, i.e., $\theta = 90^\circ$. Indeed for $Re < 125$ and $\theta = 90^\circ$ the wake has two symmetry planes (purple region in FIG. 3). The onset of unsteadiness for all yaw angles is characterized by the emergence of a unique frequency, therefore those regimes are called periodic (green, red and brown regions in FIG. 3). Except for $\theta = 60^\circ$, the onset of periodicity as a function of Re is characterized by the appearance of one reflectional symmetry plane. The brown region in FIG. 3 is very specific in the sense that it is periodic but without a clear wake symmetry. It is called quasi-symmetric since each wake structure shed is almost parallel and its centroid follows a straight line. For the highest Re and θ , symmetry and periodicity are completely lost and three-dimensional structures manifest in the wake (cyan region in FIG. 3).

We also found three main wake patterns: standing eddy, longitudinal vortex pairs and hairpin shaped vortices. Hence to summarize, if we include wake pattern in the classification of regimes, we have found 10 regimes identified as symbols in FIG. 3. In the following most of the regimes and their associated transitions are detailed and discussed. As pointed out in *Section 1*, regimes for $\theta = 90^\circ$ and $\theta = 0^\circ$ are well documented in the literature, while regimes for $60^\circ \leq \theta < 70^\circ$ do not show any features markedly different from other regimes presented in this section. For the sake of conciseness, the interested reader will hence find the detailed analysis of these regimes in *Appendix C 1*, *C 3* and *C 2*, respectively. In order to make the discussion as clear as possible, we discuss the flow regime map based on increasing Re for decreasing ranges or specific values of θ .

3.1. $\theta = 90^\circ$

The regimes associated with the flow past a $L/D = 3$ cylinder oriented perpendicularly to the flow direction are presented in *Appendix C 1* since this problem has been studied previously by Inoue and Sakuragi [28]. Nevertheless several new comments can be made with respect to the original study of Inoue and Sakuragi [28] and are summarized as follows. Up to $Re \approx 110$ (FIG. 3, \square), the flow structure past the solid body oriented perpendicularly to the flow direction presents two steady symmetric counter-rotating vortex pairs. The wake

has two symmetry planes, one vertical and one horizontal. A first unsteady bifurcation occurs at $Re = 125$. Two pairs of counter-rotating vortices are still observed in the wake but their magnitudes oscillate periodically. This regime was not observed by Inoue and Sakuragi [28]. It appears that this is a transitional since the magnitude of the lift force oscillations keeps increasing even after a computational time of $t = 500$. Increasing Re , double-sided hairpin shaped vortices are shed (*Appendix C1*, Inoue and Sakuragi [28]). The Strouhal number in this regime slowly increases as a function of Re to reach $St_z \approx 0.12$ for $Re = 175$. Our results agree perfectly well with the results of Inoue and Sakuragi [28]. In fact, Inoue and Sakuragi [28] also observed two pairs of counter-rotating vortices for $Re = 100$ and symmetric double-sided hairpin vortex shedding for $Re = 150$. With further increase of Re , other frequencies appear but the wake maintains a (x, z) symmetry plane. This symmetry plane is definitively broken with the appearance of a mode along the y direction at $Re \approx 200$. This mode generates non-negligible oscillations of the lift force along y and torque along z with the same dimensionless frequency $St_y \approx 0.05$.

In the following we briefly compare the transition scenario with another bluff body wake : the cube wake. Transitions of the cube wake have been studied by Saha [50]. Both cylinder and cube have a trivial state with two perpendicular symmetry planes. The cube wake keeps two perpendicular planes of symmetry up to $Re = 216$. Beyond $Re = 216$, the wake undergoes a steady bifurcation and loses one symmetry plane. This bifurcation was not found in the present study but could be identified by selecting additional Reynolds numbers in the range [100:125] [Note3]. The cube wake experiences a Hopf bifurcation at $Re = 270$ and the flow becomes unsteady but maintains its symmetry plane. In contrast with the cylinder wake, single-sided hairpin shaped vortices are shed periodically. Though the trivial state between the cube and cylinder wake matches, the transition sequence differs noticeably. Moreover the Reynolds number for which the transition appears differs considerably.

3.2. $70^\circ \leq \theta < 90^\circ$

Two pairs of steady counter-rotating vortices For $\theta \in [70^\circ; 90^\circ[$ and for $Re \in [25; 125]$ (FIG. 3, +), we observe two steady pairs of counter-rotating vortices past the yawed cylinder[Note4]. Unlike the $\theta = 90^\circ$ cases, the horizontal symmetry is intrinsically broken by the body inclination. The bifurcation associated with this symmetry loss appears

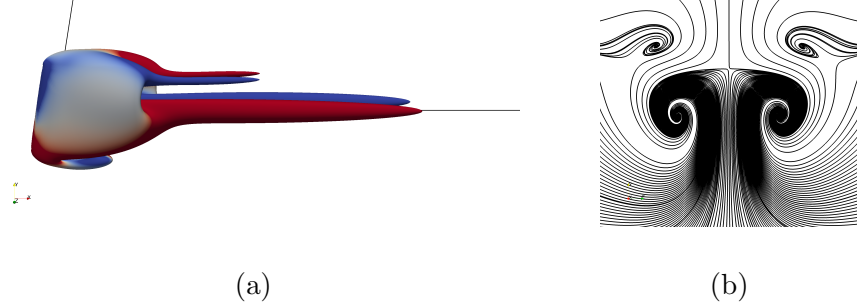


FIG. 4: Two pairs of asymmetric counter-rotating vortices at ($\theta = 75^\circ$, $Re = 100$). (a) $Q = 0.01$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction. (b) Streamlines in a (y, z) plane perpendicular to the x streamwise direction located at $5D$ from the x coordinate of the cylinder mass centre.

to be regular without hysteresis. The two pairs of vortices still have an opposite rotational direction but their intensity and shape differ as illustrated in FIG. 4a. The magnitude of the top vortex pair is lower than the magnitude of the bottom vortex pair. In this θ range, the difference of magnitude between the two pairs increases with decreasing θ . The asymmetry is also identifiable in FIG. 4b. The streamlines obtained using a projected velocity field in a plane perpendicular to the streamwise direction located a few diameters downstream of the cylinder are shown. The rotational direction of each vortex is alternated. The flow structure induces a non-zero value of the lift force in the y direction.

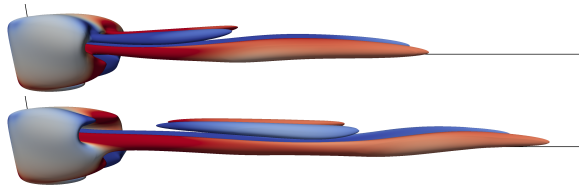


FIG. 5: Periodic shedding of two asymmetric counter-rotating vortex pairs at ($\theta = 80^\circ$, $Re = 150$). The time interval between two frames is $\Delta t = 10$. $Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

Periodic shedding of two counter-rotating vortex pairs For $Re > 125$, an unsteady bifurcation occurs. The unsteady regime is characterized by one single frequency and it is thus

called periodic. The onset of unsteadiness with respect to Re is delayed by the horizontal symmetry loss. Indeed, while the first unsteady regime appears for $\theta \approx 90^\circ$ at $Re = 125$, it only appears at $Re \approx 160$ for $\theta = 85^\circ$. For smaller yaw angles $\theta < 85^\circ$, the critical Reynolds number related to the appearance of unsteadiness decreases with decreasing θ (FIG. 3). As shown in the two snapshots in FIG. 5 for $(\theta = 80^\circ, Re = 150)$, the wake structure oscillates. The top pair of vortices has a smaller magnitude than the bottom pair of vortices. During the whole sequence, one bottom pair of vortices and one top pair of vortices are shed. The intensity of the longitudinal vorticity oscillates and consequently the vortex filaments visually disappear depending on the Q criterion value. This regime is here called periodic shedding of two counter-rotating vortex pairs (FIG. 3, Δ). We characterize this regime as transitional since its range of existence in the (Re, θ) diagram is narrow (FIG. 3). However in comparison with the periodic shedding of two counter-rotating vortex pairs with (x, y) symmetry plane observed for $\theta = 90^\circ$, the lift oscillations amplitude reaches a constant value after $t = 150$.

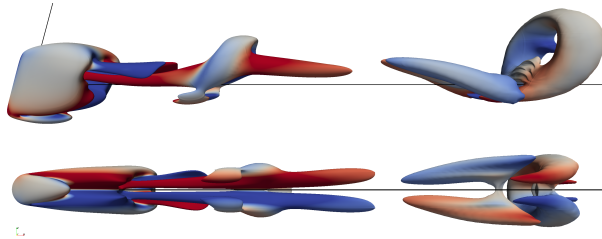


FIG. 6: Periodic shedding of single-sided hairpin vortices at $(\theta = 75^\circ, Re = 150)$.

$Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 .

The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction. Top: view in the (x, y) plane, bottom : view in the (x, z) plane, from bottom.

Periodic shedding of single-sided hairpins vortices Right after the steady-periodic transition, i.e., for slightly lower θ or slightly larger Re , we observe the shedding of single-sided hairpin vortices (FIG. 3, \blacktriangle). The hairpin vortices are shed in the y direction and have the same orientation, as illustrated in FIG. 6 for $(\theta = 75^\circ, Re = 150)$. The vertical symmetry is preserved. This regime is called single-sided since, contrary to the double-sided regime, the hairpin vortices shed periodically have always the same rotational direction. The periodic shedding of single-sided hairpin vortices is also observed for $L/D = 1$ and $(\theta = 0^\circ, Re = 360)$,

and in the case of the flow past a sphere for $Re \in [210; 300]$ [51].

A mechanism for the transition between the periodic shedding of two counter-rotating vortex pairs and the periodic shedding of single-sided hairpin vortices is presented here. One can notice that the upper pair of counter-rotating vortices of small amplitude observed in FIG. 5 is still present just above the hairpin shaped vortex nearest to the cylinder and is still attached to the farthest hairpin in FIG. 6. The lowest pair of FIG. 5 of strong amplitude gives rise to the hairpin shaped vortex of FIG. 6. This mechanism can be more clearly understood if we connect together the two hairpins vortices of FIG. 6. This connection relates to the fact that these two hairpins have the same rotational direction. Doing so we obtain a pair of streamwise vortices with (x, y) symmetry. The mechanism of transition between these two regimes shares some similarities with the mechanism of transition between the two pairs of symmetric counter-rotating vortices and the symmetric shedding of double-sided hairpin shaped vortices for $\theta = 90^\circ$ [28].

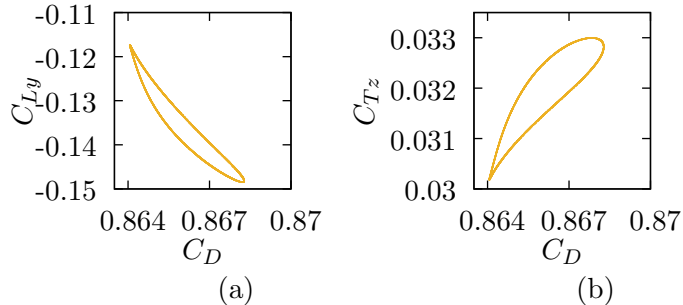


FIG. 7: Force and torque coefficient diagrams for the periodic shedding of single-sided hairpin vortices at $(\theta = 75^\circ, Re = 150)$: (a) $C_D - C_{Ly}$ and (b) $C_D - C_{Tz}$.

The $C_D - C_{Ly}$ diagram of this regime shown in FIG. 7a displays a single loop with a pointed extremity at the left top corner. This "pointed-like" behaviour increases with Re (at $Re = 135$, we observe a simple loop and no pointed extremity) and is observed for all the cases of shedding of single-sided hairpin vortices. The loop travels anticlockwise. The maximum of C_{Ly} happens slightly before the maximum of C_D (not easy to see in FIG. 7a) meaning that the signals are almost in phase. The $C_D - C_{Tz}$ diagram shown in FIG. 7b also reveals a single loop with a pointed extremity. The shift in time between the maximum of C_{Tz} and the maximum of C_D is more visible on the top right corner of the loop. Force and torque are governed by a unique dimensionless frequency $St_y \approx 0.048$.

Unsteady shedding of asymmetric double-sided hairpin vortices The value of Re corresponding to the transition between planar symmetry regimes and fully 3D regimes does not vary much when $70^\circ \leq \theta < 90^\circ$. Indeed, the transition always occurs around $Re \approx 175$. However, the 3D wake pattern noticeably changes with Re and θ . This 3D regime region is characterized by the emergence of multiple frequencies in the wake. Two different cases are chosen to illustrate the different behaviours observed: ($\theta = 85^\circ, Re = 175$) which is near the transition and ($\theta = 75^\circ, Re = 200$) that exhibits a fully developed unsteady 3D wake.

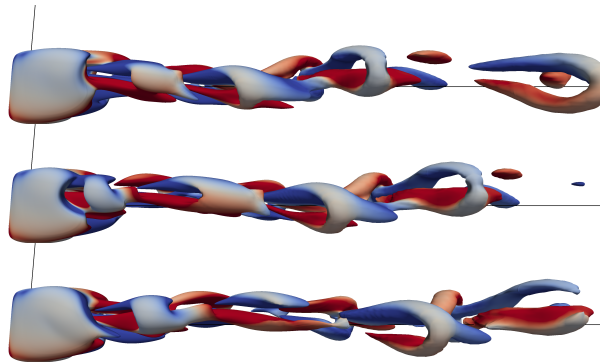


FIG. 8: Unsteady shedding of asymmetric double-sided hairpin vortices at ($\theta = 85^\circ, Re = 175$). The time interval between two frames is $\Delta t = 5$. $Q = 0.008$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

For ($\theta = 85^\circ, Re = 175$) double-sided hairpin shaped vortices are visible in the wake in FIG. 8. We observe that some hairpin vortices are slightly tilted with respect to others. Moreover the centre of the hairpin structures oscillates in the vertical direction when moving away from the cylinder. Since the wavelength of this oscillation is visually close to the one observed in the case of shedding of single-sided hairpin shaped vortices (for lower Re), we can predict that the mode along y is still present and its frequency is of the same order as for lower Reynolds numbers.

According to the $C_D - C_{Lz}$ force diagrams shown in FIG. 9a, the attractor looks like a butterfly. A DFT analysis of C_{Lz} gives a leading dimensionless frequency $St_z \approx 0.071$ and a low dimensionless frequency $St_{z,\text{low}} \approx 0.022$ of smaller amplitude. The DFT analysis of C_D gives $St_x \approx 0.047$. These three non-harmonic frequencies explain the complicated but still identifiable interacting modes in the $C_D - C_{Lz}$ diagram. The $C_D - C_{Ly}$ (respectively

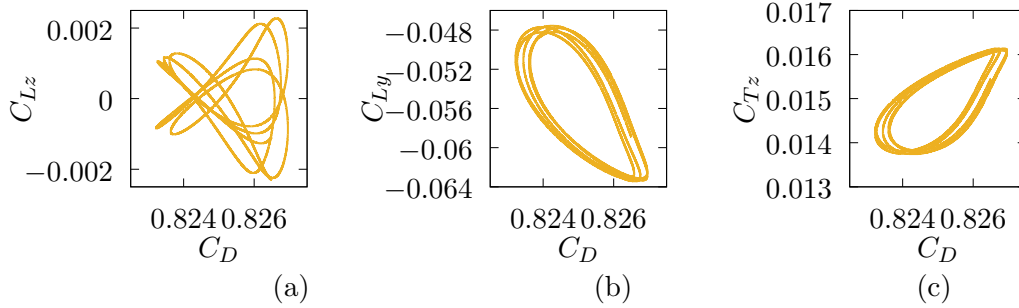


FIG. 9: Force and torque coefficient diagrams for the 3D unsteady regime dominated by the shedding of asymmetric double-sided hairpin vortices at $(\theta = 85^\circ, Re = 175)$.

$C_D - C_{Tz}$ force diagram plotted in FIG. 9b (respectively c) shows a quasi-periodic attractor with a non-zero mean value of the lift force (respectively torque). Moreover C_D and C_{Ly} are almost in phase since the extrema of both occur at the same time. C_{Lz} is much smaller than C_{Ly} , which indicates that the symmetry breaking and associated frequencies do not significantly impact the dynamics in the vertical direction. To conclude, the wake pattern and the force/torque diagrams for $(\theta = 85^\circ, Re = 175)$ indicate that this regime can be summarized as a superposition of two unsteady modes: double-sided hairpin shedding in the z direction and an asymmetric pattern in the y direction that follows from the shedding of single-sided hairpin vortices observed at lower Reynolds numbers.

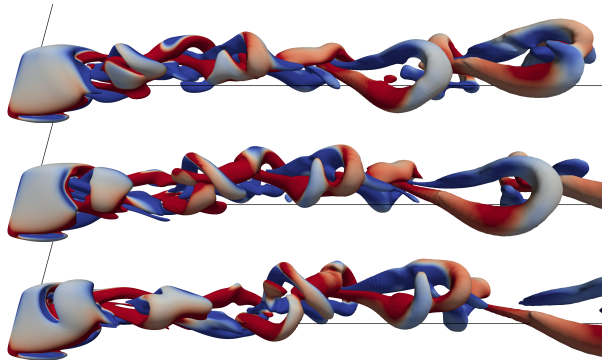


FIG. 10: Unsteady shedding of asymmetric double-sided hairpin vortices at $(\theta = 85^\circ, Re = 175)$. The time interval between two frames is $\Delta t = 5$. $Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

FIG. 10 displays the wake structure for $(\theta = 75^\circ, Re = 200)$. In comparison to $(\theta =$

$85^\circ, Re = 175$), vortices downstream of the body in the near wake and the far wake appear more interlaced and chaotic. The vortex inclination (angle between the axis belonging to the symmetry plane of each vortex and passing through its extremity, and the horizontal axis passing through the body centre) is also more pronounced.

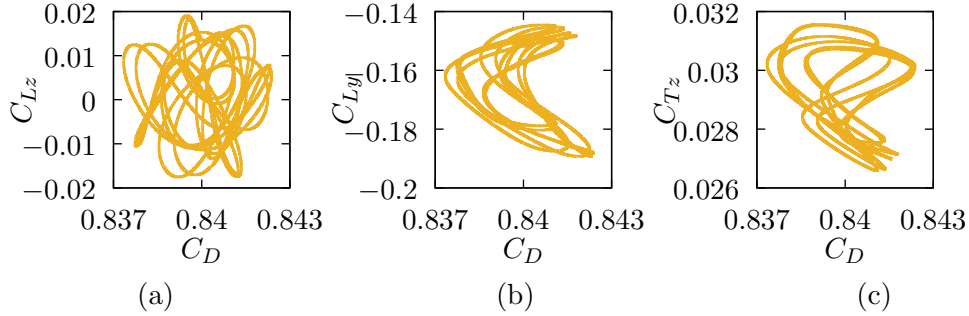


FIG. 11: Force and torque coefficient diagrams for the 3D unsteady regime dominated by the shedding of asymmetric double-sided hairpin vortices at $(\theta = 75^\circ, Re = 200)$.

The chaotic nature of the flow is markedly visible in the force/torque diagrams presented in FIG. 11a-c. The $C_D - C_{Lz}$ diagram plotted in FIG. 11a shows a kind of wool ball shape while the $C_D - C_{Ly}$ and $C_D - C_{Tz}$ diagrams show a more distinct signal that looks like a disturbed height shape. These shapes differ significantly from the shapes observed for $(\theta = 85^\circ, Re = 175)$. The magnitude of the C_{Lz} and C_{Ly} coefficients are about one order of magnitude larger than for $(\theta = 85^\circ, Re = 175)$. The C_{Lz} oscillations amplitude is similar to the C_{Ly} oscillations amplitude. Therefore the (x, y) symmetry plane is distinctly broken. The mean value of C_{Lz} approaches zero, i.e., $\overline{C_{Lz}} \simeq 0$. The dimensionless frequency of the y and z lift force oscillations are $St_y = 0.0516$ and $St_z = 0.1288$, respectively. At least two vortices are shed during one oscillation period in the y direction. St_y for $(\theta = 85^\circ, Re = 200)$ is close to St_y for $(\theta = 85^\circ, Re \leq 175)$. As for $(\theta = 85^\circ, Re = 175)$, and despite its apparent chaotic nature, the case $(\theta = 75, Re = 200)$ can be seen as the superposition of two unsteady modes : one along y and the other along z . The chaotic interaction between these perpendicular modes is representative of the cyan coloured region in FIG. 3 called 3D unsteady regime.

Evolution of the Strouhal number FIGS. 12a and b present the Strouhal numbers for $65^\circ \leq \theta \leq 90^\circ$ and $150 \leq Re \leq 250$ [Note5], respectively. In these ranges 4 unsteady regimes occur: periodic shedding of one or two pairs of counter-rotating vortices, periodic shedding of single-sided hairpin vortices and unsteady asymmetric shedding of double-sided

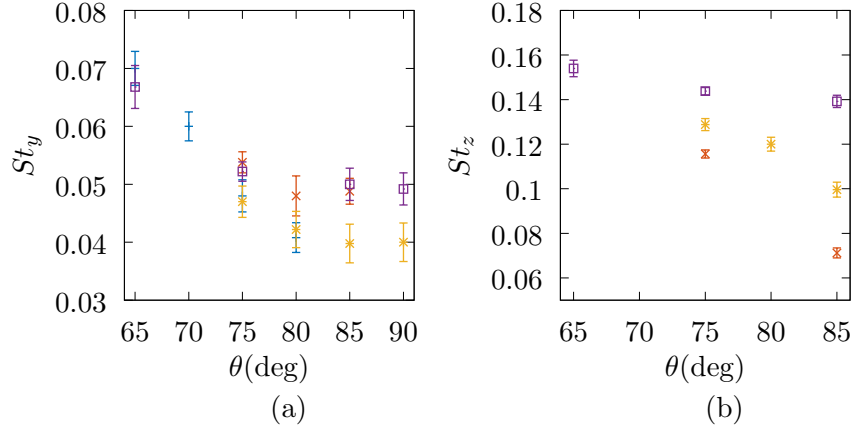


FIG. 12: Strouhal number St as a function of the yaw angle θ at $Re = 150$ (+), $Re = 175$ (\times), $Re = 200$ (*) and $Re = 250$ (\square): (a) oscillations in the y vertical direction, (b) oscillations in the z horizontal direction. The meaning of the error bars can be found in *Section 2.2.1*.

vortices. For all Re studied and for $\theta \leq 80^\circ$, St_y exhibits a decreasing trend when θ increases as shown in FIG. 12a. For larger values of θ , St_y remains constant. We do not observe a significant influence of Re on St_y . St_z decreases with the increase of θ for $\theta \in [65^\circ, 85^\circ]$, as shown in FIG. 12b. This interesting trend is contradictory to what the IP predicts, where St_z is supposed to behave as $\sin \theta$ and hence to constantly increase [47]. The possible reason explaining this discrepancy is that $L/D = 3$ is not sufficiently large to apply the IP theory. For $\theta \in [75^\circ, 90^\circ]$ and $Re \in [175, 225]$, St_z increases strongly with Re .

Comments on the $\theta = 70^\circ$ case So far, we have not discussed the case of the yaw angle $\theta = 70^\circ$. Indeed this yaw angle and its associated regimes are at the transition between regimes observed for $70^\circ \leq \theta < 90^\circ$ and $60^\circ \leq \theta < 70^\circ$. At this stage, we can say the following. For $\theta \simeq 70^\circ$, the single-sided hairpin vortex shedding regime is the first unsteady regime encountered after the steady pair of counter-rotating vortices regime (detailed in the next section). A more refined analysis in the range $Re \in [125; 135]$ might show an intermediate regime corresponding to the periodic shedding of one pair of counter-rotating vortices (also detailed in the next section). Nevertheless, we have decided to classify this specific yaw angle $\theta = 70^\circ$ in the $70^\circ \leq \theta < 90^\circ$ category as most transitions observed for $70^\circ \leq \theta < 90^\circ$ do also occur for $\theta \simeq 70^\circ$ and in particular the transition between single-sided hairpin vortices and unsteady shedding of double-sided hairpin vortices.

3.3. $60^\circ \leq \theta < 70^\circ$

For $60^\circ \leq \theta < 70^\circ$ and $Re \lesssim 125$, the steady state solution with a vertical symmetry plane changes markedly as illustrated in FIG. 3. Indeed for $\theta > 65^\circ$ we preferentially found double pairs of counter-rotating vortices while for $\theta \approx 60^\circ$ we found a single pair of counter-rotating vortices. No hysteresis is found between these two regimes: the extra vortex pair, visualized with the Q criterion, slowly vanishes as θ decreases. No symmetry-breaking and no loss of steadiness are associated with this transition. Therefore this transition is not a real bifurcation (or an interaction with an additional unstable mode) but only a continuous and smooth evolution of the wake pattern. Since the single pair of counter-rotating vortices regime is very similar to the two pairs of counter-rotating vortices regime, its description can be found in *Appendix C 2*.

The critical Reynolds number for onset of unsteadiness increases markedly when decreasing θ for $60^\circ \leq \theta < 70^\circ$ as depicted in FIG. 3. For $Re \in [125; 185]$, depending on θ , we observe two different transitions to unsteadiness. For $\theta = 65^\circ$, we observe a transition from steady shedding of one counter-rotating vortex pair to periodic shedding of one counter-rotating vortex pair (described in *Appendix C 2*) while for $\theta = 60^\circ$ we observe a transition from steady shedding of one counter-rotating vortex pair to quasi-symmetric shedding of double-sided vortices. Moreover, the path to the 3D unsteady regime when increasing Re is particularly complex for $\theta = 65^\circ$. With increasing Re , we found the following succession of regimes: periodic shedding of one pair of counter-rotating vortices, periodic shedding of single-sided hairpin vortices and eventually quasi-symmetric shedding of double-sided vortices. Reasoning in term of bifurcations, this region $Re \in [125; 185]$ is a battlefield of at least one steady mode and two unsteady modes. These multiple interactions and associated bifurcations are currently unclear and we would like to point out that this (Re, θ) region may reveal other solution types. In the following we focus on the periodic shedding of quasi-symmetric double-sided hairpin vortices.

Periodic shedding of quasi-symmetric double-sided hairpin vortices The transitions to the periodic shedding of quasi-symmetric double-sided hairpin vortices regime are quite unusual in the present context. Indeed the wake (steady for $\theta = 60^\circ$ or periodic for $\theta = 65^\circ$) loses entirely its vertical planar symmetry to reach a quasi-symmetric regime characterized by a zero amplitude y lift force. This unsteady regime prevails for $60^\circ \leq \theta < 65^\circ$ and

$Re > 175$.

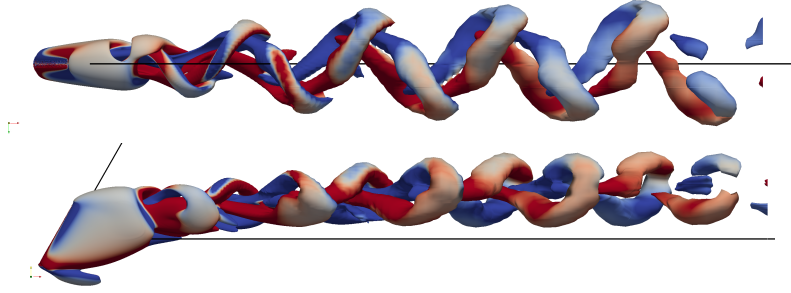


FIG. 13: The "accordion fold" regime or periodic shedding of quasi-symmetric double-sided hairpin vortices at $(\theta = 60^\circ, Re = 200)$. $Q = 0.008$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

FIG. 13 shows two perpendicular views of the typical wake pattern observed for $60^\circ \leq \theta < 65^\circ$ and $Re > 175$, i.e., in the brown region in FIG. 3. The 3D vortex structures are well organized even if no symmetry plane can be rigorously defined. The hairpins shed from both sides of the cylinder are parallel. Moreover the hairpin cores are alternatively transported on two parallel longitudinal and rectilinear paths. This regime is thus called periodic shedding of quasi-symmetric double-sided hairpin vortices. To the authors knowledge this regime has not been observed so far.

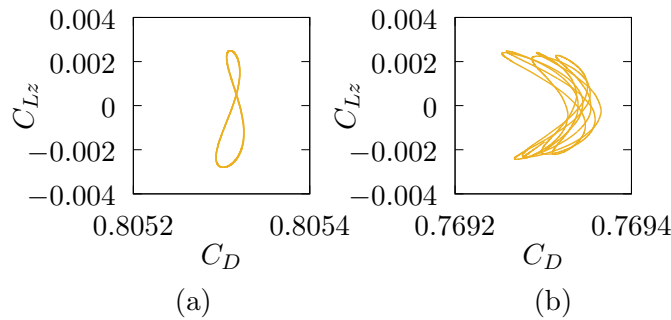


FIG. 14: $C_D - C_{Lz}$ diagrams for the periodic shedding of quasi-symmetric double-sided hairpin vortices at $\theta = 60^\circ$: (a) $Re = 200$, (b) $Re = 250$.

FIG. 14 shows the force diagrams for $\theta = 60^\circ$ and two different Reynolds numbers $Re = 200$ and $Re = 250$. The attractor for $Re = 200$ shows a "8" shape similar to the shape observed for the $(\theta = 90^\circ, Re = 150)$ shedding of double-sided vortices regime. The visible

similarity with the shedding of symmetric double-sided hairpin vortices regime justifies its "quasi-symmetric" name. The C_D oscillations amplitude is much smaller than the C_{Lz} oscillations amplitude. FIGS. 14a and b show that for both Re the amplitude of oscillations of the z lift force is similar. In this regime and for both Re , the lift force in the y direction (not shown here) has a steady non-zero value. Therefore this regime can be seen as the superposition of two perpendicular modes : one is a steady mode with a non-zero lift along y and an unsteady small amplitude mode oriented along z .

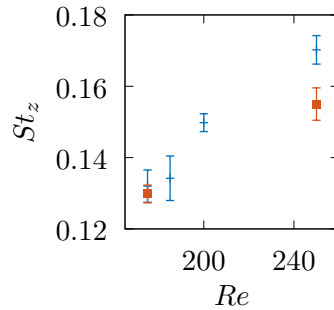


FIG. 15: St_z as a function of Re for $\theta = 60^\circ$ (+) and $\theta = 65^\circ$ (■). The meaning of the error bars can be found in *Section 22.1*.

In the region of periodic shedding of quasi-symmetric double-sided hairpin vortices, the Strouhal number increases with the Reynolds number as shown in FIG. 15. The trend followed by the computed St_z is opposite to the trend predicted by the IP. For all Re , the computed St_z decreases slightly when increasing θ . Moreover, St_z computed in this regime is close to St_z computed in other flow regimes where double-sided hairpin vortices are observed.

Finally, the wake pattern is quite unusual and is (presumably improperly) renamed the 'accordion fold' regime due to the specific position of the hairpin vortices. In the flow map shown in FIG. 3, the 'accordion fold' regime is surrounded by 3D unsteady and single-sided regimes. This raises questions about how unstable modes grow and compete with each other. A small change of the cylinder tilt angle θ by a few degrees drastically changes the symmetry/lack of symmetry and associated wake patterns. This question could be further elucidated by the bifurcation theory [14, 18].

3.4. $\theta = 55^\circ$

For $\theta = 55^\circ$ the onset of unsteadiness is characterized by the transition from steady shedding of two vortex pairs to shedding of single-sided hairpin vortices. The transition between these two regimes has been described before and we only make short additional comments below.

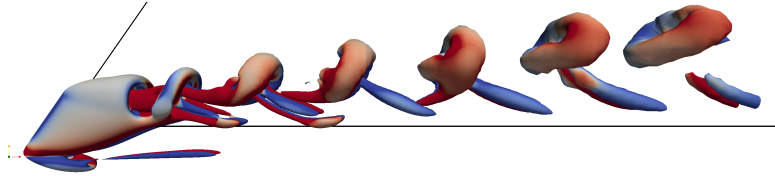


FIG. 16: Original periodic shedding of single-sided hairpin vortices at ($\theta = 55^\circ, Re = 250$). $Q = 0.008$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

FIG. 16 displays another original wake pattern for ($\theta = 55^\circ, Re = 250$) that exhibits single-sided hairpin vortices with vertical planar symmetry. However some differences are noticeable in comparison with the shedding of single-sided hairpin vortices observed for $70^\circ \leq \theta \leq 85^\circ$. The size of vortices is visibly smaller in FIG. 16 than in FIG. 7. Hairpin vortices are shed not only along x but also along y , since the vortices monotonously move away from the x axis in the cylinder wake. Furthermore the hairpin vortices nearest to the cylinder are connected between each other by longitudinal vortices. We also observe a pair of counter-rotating vortices at the bottom of the cylinder that was not present for $70^\circ \leq \theta \leq 85^\circ$. Finally the dimensionless vortex shedding frequency is $St_y \approx 0.17$. This value is at least twice larger than the largest dimensionless frequency obtained in the range $70^\circ \leq \theta \leq 85^\circ$ and relates to the larger number of vortices observed in FIG. 16 compared to FIG. 7. The $C_D - C_{Lz}$ diagram shows a simple closed loop of very small amplitude (not shown here) as observed in the range $70^\circ \leq \theta \leq 85^\circ$.

3.5. $0^\circ < \theta < 55^\circ$

The steady shedding of one or two pairs of counter-rotating vortices are the two wake patterns observed for $\theta = 0^\circ$ up to $\theta \approx 55^\circ$ for all Re . These regimes have been documented

in *Appendix C 2* and *Section 3 3.2*. No periodic regime is observed and the vertical planar symmetry is always present. As shown in FIG. 3, the one pair regime prevails for $Re \leq 150$ while the two pair regime increasingly dominates for higher Re . This behaviour suggests that the appearance of the second pair of vortices is linked to the amount of vorticity produced at the surface of the cylinder that scales as $Re^{1/2}$ following the boundary layer theory.

3.6. $\theta = 0^\circ$

When the cylinder is aligned with the flow direction a steady toroidal vortex is observed (the reader is referred to *Appendix C 3* for more details).

3.7. Discussion

Additional comments on the aspect ratio The flow map shown in FIG. 3 and the analysis of the flow regimes that we have performed so far reveal the strong dependence of the wake pattern on both Re and θ . We would like to shortly comment here on the transition between some regimes in light of results available in the literature. The results of Ramberg [47] for long cylinders ($L/D \geq 20$, $160 \leq Re \leq 1000$) (FIG. 2c in his article) are valuable data to discuss similarities and differences with our own results presented in this work. Ramberg [47] observed two different regimes: (i) a steady regime with one counter-rotating vortex pair for low θ and (ii) a periodic shedding of oblique vortices for high θ (Von-Kármán vortex streets). The transition between these two regimes occurs approximatively at $\theta = 40^\circ$ and decreases with Re . There are two main differences between the map drawn by Ramberg [47] and ours. Firstly, he observed oblique vortex shedding for high θ , while we observe hairpin vortex shedding. Secondly, the transition between the regimes observed by Ramberg [47] and the transition between our regimes does not occur for the same Re and θ . These differences are attributed to different aspect ratios in Ramberg [47]’s study and ours. Inoue and Sakuragi [28], when studying the flow past a cylinder in a cross flow, also pointed out that for a fixed Reynolds number the aspect ratio has a significant impact on the wake pattern. For instance, at $Re = 150$, hairpin vortex shedding is observed for a $L/D = 3$ cylinder while oblique vortex shedding is observed for a $L/D = 25$ cylinder.

In the opposite limit of small aspect ratios ($L/D = 1/50$) Gao et al. [20] studied nu-

merically the effect of the yaw angle on the wake. They distinguished five different states : steady state, periodic state, periodic state with a low frequency modulation, quasi-periodic state and chaotic state. As for the cylinder aligned with the flow direction ($\theta = 0^\circ$) a small inclination results in a loss of axisymmetry. They also observed that the bifurcation sequence occurring for $\theta = 0^\circ$ is generally observed for other angles. If we use their convention, we can draw the same conclusions: the first unsteady regime observed when increasing Re is always periodic and it is then disturbed by one or more frequencies. Nevertheless a distinctive feature between a disk and a cylinder can be underlined. In the disk case the periodic regimes are most of the time (except in the region of arbitrary symmetry plane as discussed in Gao et al. [20]) characterized by a (x, y) symmetry plane while in the cylinder case (x, y) symmetry is encountered but also (x, z) quasi-symmetry as detailed in *Section 3.3.3*.

Comparison with the freely falling cylinder Toupoint et al. [55] experimentally investigated the fall of cylinders of density close to the fluid density for $2 \leq L/D \leq 20$ and various Reynolds numbers. For $L/D = 3$ and $Re \approx 120$, where the Reynolds number is based on the terminal vertical velocity, they observed a steady and rectilinear fall. They also identified two pairs of counter rotating vortices in the cylinder wake in agreement with the results of Inoue and Sakuragi [28] and the present results. For higher Reynolds numbers ($Re \approx 270$) a second regime is observed named irregular 2 in their paper. This regime is characterized, among other features, by oscillations of the cylinder inclination angle in the range $80^\circ \lesssim \theta \lesssim 90^\circ$ as it falls. The main frequency associated with this fluttering motion is $St \approx 0.05$. Furthermore the cylinder undergoes oscillations of its velocity in the gravity direction with a frequency $St_x \approx 0.2$. We have seen in *Section 3.3.1* that the cylinder undergoes strong torque oscillations in z direction. The associated frequency is 0.05. This frequency is very close to the frequency encountered during the fluttering motion. The loss of planar symmetry in the $\theta = 90^\circ$ fixed case might be the root of the apparition of fluttering motion in the case of free fall. However the drag oscillations, i.e., C_x oscillations in the present configuration, for $Re = 250$ have a dominant dimensionless frequency smaller than the frequency encountered during the free falling case. Despite some similarities the two problems are thus fundamentally different. The inclined body case can be considered as the limit where the density of the particles is infinite, a limit which is far from the density ratio of Toupoint et al. [55].

4. TIME AVERAGED RESULTS : LOAD ON THE CYLINDER

We discuss here the time averaged load on the cylinder as a step to derive the closures needed in the averaged momentum equations [9]. When the flow is steady (blue, yellow and purple areas of FIG. 3), the averaged loads are simply equal to the instantaneous loads. The results of the previous section show that the modes disrupting the (x, y) symmetry plane have a negligible signature on the loads. For instance the C_{Lz} oscillations amplitude for $Re = 200$ and $\theta = 75^\circ$ (FIG. 11) only amounts to 5% of the mean drag coefficient and is at least one order of magnitude smaller than the y oscillations amplitude. The modes disrupting the (x, z) symmetry plane are more energetic and create non-negligible force and torque oscillations. Indeed FIG. 9 (b) shows that the y force oscillations amplitude reaches a value approximatively equal to 25% of the mean value.

In this section we suggest simple laws for the drag force, the lift force and the hydrodynamic torque as a function of θ and Re . We first shortly review different approaches to this problem from the literature and then suggest simple laws for C_D , C_{Ly} and C_{Tz} . We then put the suggested laws to the test by comparing their performance to our computed results. In order to show the effect of the unsteady modes on the loads, the mean loads and the mean amplitude of the loads are shown (with errorbars).

4.1. State of the art

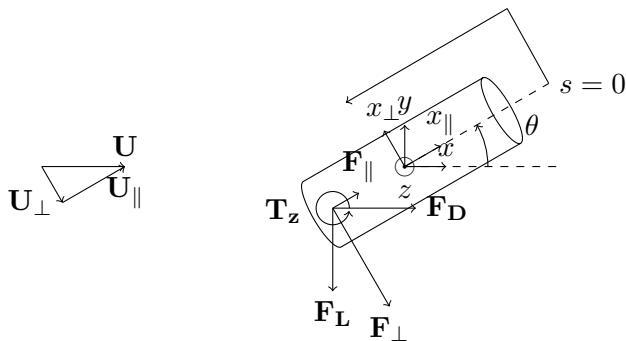


FIG. 17: Sketch of the force and torque exerted on a yawed cylinder.

The force and torque experienced by a finite-length cylinder in a steady flow are not known exactly even in the Stokes flow regime $Re = 0$. However, owing to the linearity of the

equations in this regime, the force exerted on a cylinder tilted with an angle θ (see FIG. 17) can be related to the force exerted on the same cylinder tilted by $\theta = 0^\circ$ and $\theta = 90^\circ$ as:

$$C_{\perp} = C_{\perp\theta=90^\circ} \sin \theta \quad (2)$$

$$C_{\parallel} = C_{\parallel\theta=0^\circ} \cos \theta \quad (3)$$

where C_{\perp} and C_{\parallel} are the normal force and parallel force coefficients, respectively. The drag and lift coefficients follow as:

$$C_D = C_{D\theta=0^\circ} \cos^2 \theta + C_{D\theta=90^\circ} \sin^2 \theta \quad (4)$$

$$C_{Ly} = C_{D\theta=0^\circ} \cos \theta \sin \theta - C_{D\theta=90^\circ} \sin \theta \cos \theta \quad (5)$$

In the Stokes regime, the hydrodynamic torque on a cylinder is zero : a freely moving cylinder keeps its initial orientation while falling under gravity. This specific property is lost when including weak effects of inertia [13]. When $Re \ll 1$, the torque coefficient along z varies as :

$$C_{Tz} \propto \sin \theta \cos \theta \quad (6)$$

For high Reynolds numbers, the IP states that the normal force on an infinitely long yawed cylinder in a flow of velocity U is the same as the force exerted on the same cylinder placed in a cross flow of velocity $U \sin \theta$ [53]. The normal coefficient can thus be written as [23]:

$$C_{\perp} = C_{\perp\theta=90^\circ} \sin^2 \theta \quad (7)$$

and the drag and lift coefficients as:

$$C_D = C_{D\theta=90^\circ} \sin^3 \theta \quad (8)$$

$$C_{Ly} = C_{D\theta=90^\circ} \sin^2 \theta \cos \theta \quad (9)$$

In order to account for the force experienced by a cylinder aligned with the flow direction into the drag law, Rosendahl [49] proposed the following empirical relation based on the IP:

$$C_D = C_{D\theta=0^\circ} + (C_{D\theta=90^\circ} - C_{D\theta=0^\circ}) \sin^3 \theta \quad (10)$$

There are several other empirical and semi-empirical laws derived for the drag force exerted on non spherical-particles as, e.g., the law suggested by [24]. In [24], the authors

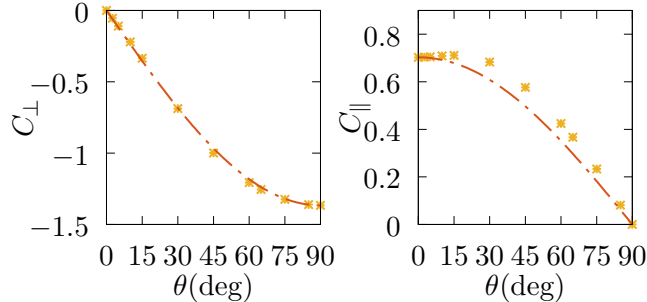


FIG. 18: Perpendicular and parallel force coefficients for $Re = 50$: * numerical results,
 --- (2) and (3).

proposed to use the lengthwise and crosswise sphericity (whose definition can be found in their article) instead of using explicitly the orientation of the particle. Their correlation gives good agreement with existing results in the literature when dealing with general complex shape particles. Here we are interested in cylindrical particles and attempt to develop a specific law for such a body shape.

4.2. Simple laws for the hydrodynamic force and torque

One of the drawbacks of the laws proposed above is that, for most of them, they explicitly depend on the drag force exerted on the cylinder when it is aligned with the flow direction, i.e., $C_{D\theta=0^\circ}$. The flow past a cylinder perpendicular to the flow direction has been studied extensively and a precise correlation has been proposed by Clift et al. [11, p. 154]. This is not the case when the cylinder is aligned with the flow direction. Therefore, for practical purposes it is desirable to find a law that does not involve $C_{D\theta=0^\circ}$ explicitly. Moreover, the validity of the laws proposed in the previous subsection as a function of Re has not been assessed for the present range of $Re \in [25, 250]$. Our intention here is to provide simple laws by combining our own numerical results and results from the literature.

Sanjeevi and Padding [52] studied recently the flow past oblate spheroids. They observed that equation (4) obtained in the Stokes regime matches well numerical results even in inertial configurations ($Re = 2000$). Even if cylinders have a shape different from spheroids we may expect the same behaviour for cylinders and therefore a good agreement of equation (4) with our numerical results even in inertial regimes. This is verified in FIG. 18. The forces perpendicular and parallel to the flow direction fit very well equations (2) and (3)

obtained in the Stokes regime. We note a very slight difference between equation (3) and our numerical results for $30^\circ \leq \theta \leq 60^\circ$.

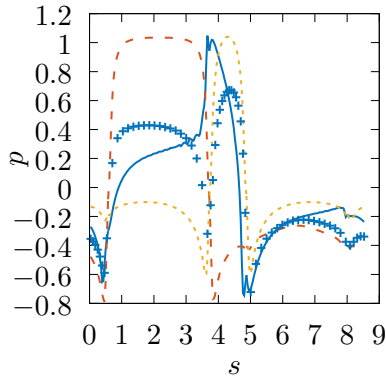


FIG. 19: Pressure as a function of the distance along the perimeter of a rectangle contained in the (x, y) plane for different value of θ . — $\theta = 90^\circ$, - - $\theta = 0^\circ$, — $\theta = 30^\circ$, + linear theory for $\theta = 30^\circ$.

FIG. 19 displays the dimensionless pressure $p/(1/2\rho U^2)$ (the reference pressure at the far field is chosen to be zero) for different values of θ as a function of s . s is the normalized distance along the perimeter of a rectangle contained in the (x, y) plane. The rectangle centre is the cylinder centre and the rectangle dimensional length and width are respectively $L + 0.15D$ and $D + 0.15D$. s is chosen to be zero downstream of the cylinder, on the cylinder symmetry line, as sketched in FIG. 17. In the Stokes regime owing to the linearity of the equations $p = p_{\theta=0^\circ} \cos \theta + p_{\theta=90^\circ} \sin \theta$. FIG. 19 shows that the linearity theory (blue crosses) does not fit our numerical results for $Re = 50$ and $\theta = 30^\circ$ (blue line). The good agreement between our numerical results and equations (2) and (3) is not due to linearity of the pressure and velocity fields. So why do the Stokes-based laws work so well even for moderate Re ? Sanjeevi and Padding [52] proposed an explanation based on a scaling of the median surface pressure $p \propto (\mathbf{U} \cdot \mathbf{n})^2$, where \mathbf{n} is the normal to the body in the median plane ((x, y) plane in our configuration). This scaling performs well in their situation but is not suited to our cylinder since \mathbf{n} in the (x, y) plane is a constant over each of the three main surfaces of the cylinder, i.e., its two end disks and its length. The good agreement between the Stokes-based laws and our present numerical results remains elusive at this stage and may be the consequence of a subtle combination of pressure and friction forces.

Equations (4) and (5) require $C_{D\theta=0^\circ}$ which is a priori unknown. Results obtained using

slender body theory ($L \gg D$) (see for instance Batchelor [6]) can be used to obtain a rough estimate of $C_{D\theta=0^\circ}$. At the lowest order in D/L the slender body theory gives [Note6]: $C_{D\theta=0^\circ} \approx 1/2 C_{D\theta=90^\circ}$. Thus for moderate Reynolds numbers $25 \leq Re \leq 100$ we suggest the following law for the drag:

$$C_D \approx C_{D\theta=90^\circ} \left(\frac{1}{2} \cos^2 \theta + \sin^2 \theta \right), \quad (11)$$

and the lift :

$$C_{Ly} \approx C_{Ly\theta=90^\circ} \left(\frac{1}{2} \cos \theta \sin \theta - \sin \theta \cos \theta \right). \quad (12)$$

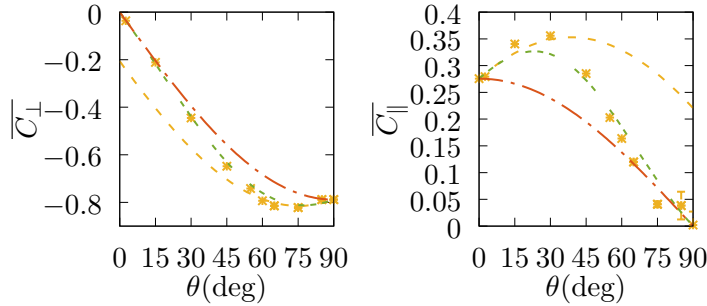


FIG. 20: Perpendicular and parallel force coefficients for $Re = 250$: $*$ numerical results, $---$ (2) and (3), $---$ (13) and (14), $---$ (15) and (16). The error bars indicate the magnitude of the unsteady modes.

As can be seen in FIG. 20, laws (2) and (3) derived in the Stokes regime are not able to reproduce accurately the drag and lift forces exerted on the cylinder at the highest Reynolds number $Re = 250$. An important deviation is observed in the range $45^\circ \leq \theta \leq 75^\circ$ for C_\perp and $15^\circ \leq \theta \leq 60^\circ$ for C_\parallel . For the former this deviation might be the result of an increase of the friction (with respect to the aligned cylinder case). Indeed in the previous section we observed that the increase of Re results in the appearance of a second pair of vortices in the steady wake. This appearance is related to the increase of vorticity production at the cylinder surface. This vorticity production explains the increase of the friction force with respect to $\theta = 0^\circ$ where no counter rotating vortices are observed. Except for $\theta > 75^\circ$ and C_\parallel , the unsteady modes have a negligible effect on the perpendicular and parallel forces. Indeed the unsteady modes perturb either or both the (x, y) and (x, z) planes but not specifically the planes perpendicular to \mathbf{x}_\perp or \mathbf{x}_\parallel . As a consequence the unsteady modes have a smaller signature on C_\parallel and C_\perp than on C_D and C_{Ly} .

We offer below simple arguments to modify the laws in order to give reasonable agreement in this regime. The basic idea is to keep the behaviour of the Stokes law when it works well (low θ for C_{\perp} and high θ for C_{\parallel}) and change the law when departure manifests. When $\theta \approx 90^{\circ}$ C_{\perp} behaves as :

$$C_{\perp} \approx C_{\perp\theta=90^{\circ}} (A \cos \theta + \sin \theta) \quad (13)$$

where A is a function of Re which can be fitted to our numerical results. Since A increases very slowly with Re we choose it as a constant $A = 0.26$. When $\theta \approx 0^{\circ}$ a similar relationship can be obtained for the force coefficient parallel to the cylinder :

$$C_{\parallel} \approx C_{\parallel\theta=0^{\circ}} (\cos \theta + B \sin \theta), \quad (14)$$

where B is a function of Re . From our numerical results, we note that B should be a slightly increasing function of Re but for the sake of simplicity B is chosen constant and equal to $B = 0.80$. As shown in FIG. 20, (13) and (14) match well our numerical results for high and low yaw angles, respectively. In order to obtain a solution uniformly valid on the whole range of θ , (13) and (14) are matched to equations (2) and (3). To this aim we use powers of trigonometric functions in order to make some terms of (13) and (14) negligible when the other terms approach the Stokes law and obtain:

$$C_{\perp} \approx C_{\perp\theta=90^{\circ}} \sin \theta + 0.26 C_{\perp\theta=90^{\circ}} \cos \theta \sin^2 \theta \quad (15)$$

$$C_{\parallel} \approx C_{\parallel\theta=0^{\circ}} \cos \theta + 0.8 C_{\parallel\theta=0^{\circ}} \sin \theta \cos^2 \theta \quad (16)$$

Even if the above equations contained terms similar to the terms in the Stokes law, they are different from equations obtained using a formal development in Re in the Oseen regime [13]. This is not surprising since they have been derived using numerical simulations for $Re \gg 1$. (15) and (16) match pretty well our numerical results, as plotted in FIG. 20. Expressions for the drag coefficient and the lift coefficient can be obtained straightforwardly but we still need $C_{D\theta=0^{\circ}}$. Based on our numerical results we choose $C_{D\theta=0^{\circ}} = 0.4 C_{D\theta=90^{\circ}}$. The resulting equations shown below are used for the highest Reynolds number $150 \leq Re \leq 250$:

$$C_D \approx C_{D\theta=90^{\circ}} [0.4 \cos \theta (\cos \theta + 0.8 \sin \theta \cos^2 \theta) + \sin \theta (\sin \theta + 0.26 \cos \theta \sin^2 \theta)], \quad (17)$$

$$C_{Ly} \approx C_{D\theta=90^\circ} [0.4 \sin \theta (\cos \theta + 0.8 \sin \theta \cos^2 \theta) - \cos \theta (\sin \theta + 0.26 \cos \theta \sin^2 \theta)]. \quad (18)$$

At this stage several comments can be made. Laws (11) and (12) obtained using linearity of Stokes equations might be valid for other aspect ratios. Indeed the linearity of Stokes equations remains valid for all aspect ratios and these laws have proven to be valid even for spheroids of high aspect ratio [52]. The assumption $C_{D\theta=0^\circ} \approx 1/2 C_{D\theta=90^\circ}$ is also supposed to give more accurate results as L/D increases [6]. However (17) and (18) have no reason to be valid for other L/D .

Deriving a law for the torque (or pitching-torque since the angular velocity is zero) valid for high Reynolds number flows and arbitrary L/D is beyond the scope of the present study. The reader interested in that question may consult Zastawny et al. [63]. However, physical arguments and a simple model are given in the following. The hydrodynamic torque reads $\mathbf{T} = \int_S \mathbf{r} \times \boldsymbol{\sigma} \cdot \mathbf{n} dS$ where \mathbf{r} is the local position relative to the cylinder mass centre. We define the hydrodynamic centre \mathbf{r}_{hc} as the point where the hydrodynamic torque vanishes. The hydrodynamic torque can thus be written:

$$\mathbf{T} = \mathbf{r}_{hc} \times \mathbf{F}. \quad (19)$$

In the following, straightforward assumptions are made to derive a closed expression for the torque. We define a Cartesian coordinate system as $(x_{\parallel}, y_{\perp}, z)$ where x_{\parallel} is the axis of symmetry of the cylinder starting from its mass centre, as illustrated in FIG. 17. Our main motivation is to find the z component of the torque. From flow symmetry considerations, we assume that $\mathbf{r}_{hc} \approx (x_{\parallel hc}, x_{\perp hc}, 0)$ which does not seem to be a strong assumption except in cases where (x, y) planar symmetry is broken. We further assume that the parallel component of the hydrodynamic centre is larger than the perpendicular component, i.e., that $\mathbf{r}_{hc} \approx (x_{\parallel hc}, 0, 0)$. This assumption is expected to be valid for long cylinders $L \gg D$. We obtain:

$$T_z \approx x_{\parallel hc} F_{\perp} \quad (20)$$

From (20) and assuming that we know the normal force exerting on the cylinder, the new unknown to derive an expression for the torque is the position of the hydrodynamic centre $x_{\parallel hc}$. This equation supports the choice of L as the characteristic length in the expression of C_{Tz} . $x_{\parallel hc}$ is made dimensionless by dividing by L rather than D . With the help of (20), (6) derived for $Re \ll 1$ can be understood as follows: since to leading order in Re the

normal force varies as $\sin \theta$, the hydrodynamic centre position $x_{\parallel hc}$ varies as $\cos \theta$. Two main questions remain to be addressed (i) is this law valid for higher Reynolds numbers ? and (ii) can we estimate the magnitude of the torque ?

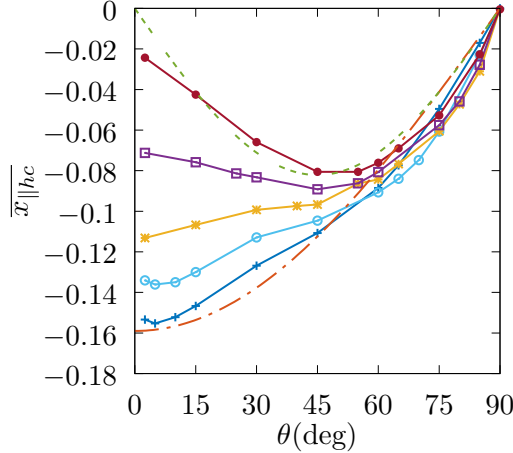


FIG. 21: Position of the hydrodynamic centre normalized by the length as a function of the yaw angle: $+$ $Re = 50$, \circ $Re = 100$, $*$ $Re = 150$, \square $Re = 200$, \bullet $Re = 250$, $---$ (21) for $Re = 50$, $---$ (22) for $Re = 250$.

FIG. 21 aims to give some answers to these questions. It plots the variation of $x_{\parallel hc}$ as a function of θ for different Re . The hydrodynamic centre position is computed as: $\overline{x_{\parallel hc}} = \overline{T_z / F_{\perp}}$ using our numerical results. This expression becomes singular when $\theta \rightarrow 0$. We note that $x_{\parallel hc}$ stays close to the cylinder mass centre for all the configurations studied. For $Re = 50$ and $Re = 100$, $x_{\parallel hc}$ varies as $\cos \theta$. Thus, as for the force, the law (6) obtained in the Oseen regime remains valid for moderate Reynolds number. Since the magnitude of $x_{\parallel hc}$ decreases linearly with Re for $\theta = 45^\circ$, we suggest the following law for the hydrodynamic centre position:

$$x_{\parallel hc} = \sqrt{2}(aRe - b) \cos \theta. \quad (21)$$

where $a = 1/6600$ and $b = 0.12$ are parameters fitted with our numerical results. This law is used for $25 \leq Re \leq 150$.

For $Re = 200$ and $Re = 250$, the magnitude of $x_{\parallel hc}$ decreases with decreasing θ for $\theta \leq 45^\circ$. (21) is not able to capture this behaviour. We thus suggest the following law:

$$x_{\parallel hc} = 2(aRe - b) \sin \theta \cos \theta, \quad (22)$$

which fits very well our numerical results for $Re = 250$. Once again some comments can be

made on the validity of these correlations. For $Re \ll 1$, calculations in the Oseen regime [13] predict a linear increase of the dimensional torque with Re and a dimensional normal force constant to leading order in Re . Thus, to leading order in Re , $x_{||hc}$ increases linearly with Re . This disagrees with (21) where the magnitude is seen to decrease with Re . One possible reason for this discrepancy is that the Reynolds numbers studied here are probably already far beyond the validity of the Oseen regime. Nevertheless this needs further investigation. We would like to point out that (22) should be taken cautiously since it tends to zero for $Re \approx 792$, which clearly disagrees with previous observations on the flow past a cylinder at high Reynolds number [29].

4.3. Numerical results

Moderate Re ($25 \leq Re \leq 100$) FIGS. 22abde exhibit a highly satisfactory agreement between the predictions obtained in the Stokes regime with (4) and (5) and our numerical results for both the lift coefficient and the drag coefficient at $Re = 25$ and $Re = 50$. The semi-empirical relations based on linear laws (11) and (12) also provide a good agreement with our numerical results. As observed by Sanjeevi and Padding [52] for spheroidal particles, the laws derived by assuming the linearity of Stokes equations match closely the numerical results even for inertial regimes. For $Re = 25$ and $Re = 50$, the predictions of both the lift coefficient and the drag coefficient given by the semi empirical relation of Rosendahl [49] are less accurate than the laws based on the linearity of Stokes equations.

As Re increases to 75 and 100, the drag force plotted in FIG. 22gh and FIG. 22jk, respectively, slowly departs from the predictions obtained in the Stokes regime. Indeed the drag coefficient is underestimated by (4) and (11), especially for $\theta \in [30^\circ, 75^\circ]$. However the predictions for the lift coefficient given by the laws derived by assuming the linearity of Stokes equations still agree very well with our numerical results.

FIG. 22c, FIG. 22f, FIG. 22i and FIG. 22l show the torque coefficient for $Re = 25$, $Re = 50$, $Re = 75$ and $Re = 100$, respectively, and compare the law (20) derived by assuming a Oseen regime (based on (21) and (2), [13]) to our numerical results. The torque is zero for $\theta = 0^\circ$ and $\theta = 90^\circ$ but the only stable configuration is $\theta = 90^\circ$ as explained by Khayat and Cox [32]. The agreement between our numerical results and (20) based on (21) and (2) is satisfactory even if our numerical results plot is slightly shifted to higher values for high

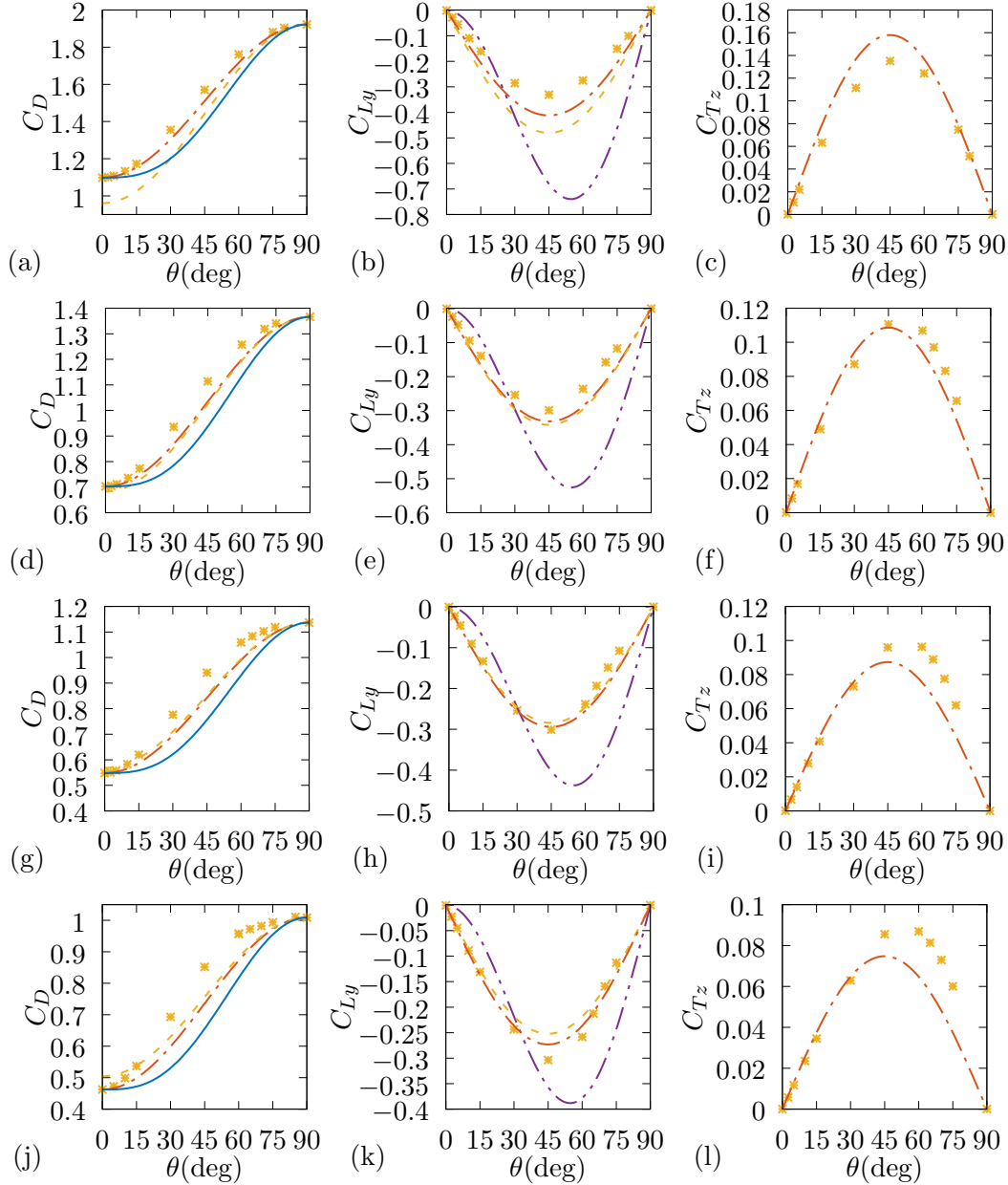


FIG. 22: Drag, lift and torque coefficients for: a-c $Re = 25$, d-f $Re = 50$, g-i $Re = 75$, j-l $Re = 100$. \ast numerical results, $-$ empirical relation (10) of Rosendahl [49], $- - -$ (4) and (5) for the force / (20) based on (21) and (2) for the torque, $- - - -$ (11) and (12).

θ when increasing Re . We also note that for $Re = 25$, (20) based on (21) and (2) slightly overestimates the maximum torque. This overestimation is attributed to an overestimation of the position of the hydrodynamic centre $x_{||hc}$. For $Re = 75$ and $Re = 100$, (20) based on (21) and (2) slightly underpredicts the maximum torque. This underestimation is here attributed to an underprediction of the normal force F_{\perp} .

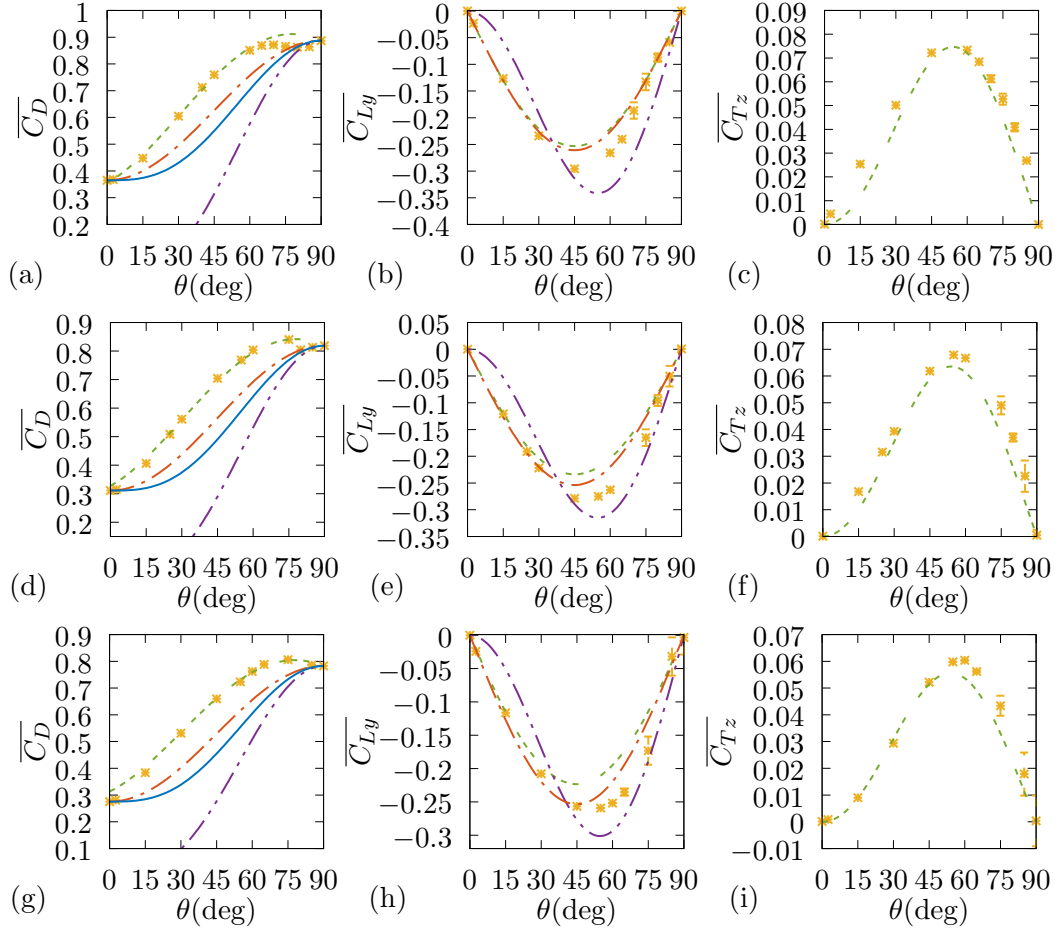


FIG. 23: Drag, lift and torque coefficients for: a-c $Re = 150$, d-f $Re = 200$, g-i $Re = 250$.

* numerical results, — empirical relation (10) of Rosendahl [49], - - - (4) and (5), - · - · - IP, - - - (17) and (18) for the force / (20) based on (22) and (15) for the torque. The error bars indicate the magnitude of the unsteady modes.

High Re ($100 < Re \leq 250$) FIGS. 23dg show that the drag force does not reach a maximum for $\theta = 90^\circ$ but for an angle close to $\theta = 75^\circ$. This behaviour is related to a scaling of the drag force with the pressure force for $Re \gg 1$. It is common in practice to relate the pressure force to the projected surface of the cylinder with respect to the inlet velocity : $S_{proj} = \pi D^2/4 \cos \theta + LD \sin \theta$. The projected surface is maximum for $\theta \approx 75.3^\circ$ which explains the observed behaviour. FIGS. 23adg clearly evidence that the IP in its original form given by (8) does not fit well our numerical results for the drag coefficient since the drag coefficient when $\theta = 0^\circ$, i.e., $C_{D\theta=0^\circ}$, is not taken into account. Taking into account $C_{D\theta=0^\circ}$ leads to (10) that exhibits a better agreement. Marked differences in

the range $30^\circ \leq \theta \leq 60^\circ$ are still noticeable though. For all θ , the linear law yields a more satisfactory match with our numerical results than the IP and its modification due to Rosendahl [49]. The law (17) based on geometric arguments is the law that gives the best agreement for the drag force for $Re = 150, 200$ and 250 . In the limit of high θ , we do note a slight departure of this law from our numerical results. For $Re = 150$ the drag coefficient is almost constant for $75^\circ \leq \theta \leq 90^\circ$ while (17) predicts a slight increase of $\overline{C_D}$ from 90° to 70° degrees.

The agreement between the IP and our numerical results is better for the lift coefficient, as illustrated in FIGS. 23beh, than for the drag coefficient. The IP gives results of comparable accuracy to (5). The law (18) based on geometric arguments slightly underpredicts the lift coefficient. This is due to the fact that (15) overpredicts the parallel force for $50^\circ \leq \theta \leq 80^\circ$.

The torque coefficient is still well approximated by (20) based on (22) and (15) even if we observe that the maximum of C_{Tz} is slightly underpredicted for the three values of $Re = 150, 200$ and 250 . This underprediction is attributed to an underprediction of the hydrodynamic centre position $x_{||hc}$.

General outcome and further comments on force and torque laws The overall picture of force and torque laws is as follows. The laws obtained in the Stokes regime or the Oseen regime for the drag and lift coefficients give a good agreement for $Re \leq 100$. At higher Reynolds numbers (17) and (18) should be preferred. The torque is well approximated by (20), provided the normal force and the hydrodynamic centre position are properly modelled as a function of Re . The value of θ for which the torque is maximal increases from 45° to 55° when Re increases.

FIGS. 22 and 23 also show that $C_{D\theta=90^\circ}$ and $C_{D\theta=0^\circ}$ decrease significantly as Re increases as $Re^{-1/2}$ (not shown here). This is in line with the scaling of the viscous stress in the boundary layer approximation [7]. However the maximum lift force as a function of θ remains approximatively constant as Re increases. This kind of behaviour has also been observed for the lift force exerted on a sphere immersed in a linear shear flow Kurose and Komori [34].

5. CONCLUSION

We investigated the inertial flow past a 3D cylinder of aspect ratio 3 by particle-resolved simulation. The ranges of Reynolds number Re and yaw angle θ examined are $[25, 250]$ and $[0^\circ, 90^\circ]$, respectively. We analysed the wake structure and the hydrodynamic force and torque exerted on the cylinder. Ten different regimes and corresponding wake patterns were observed depending on θ and Re spanning standing toroidal eddy, one or two steady counter-rotating vortex pairs, periodic shedding of single-sided hairpin vortices, quasi-symmetric shedding of double-sided vortices nicknamed accordion fold, periodic shedding of one or two pairs of counter-rotating vortices, and unsteady shedding of asymmetric double-sided hairpin vortices. When the cylinder is perpendicular to the flow direction, we confirm the transition scenario observed by Inoue and Sakuragi [28]. A slight inclination of the cylinder with respect to the perpendicular configuration modifies dramatically the steady and first unsteady states through a loss of one symmetry plane. For sufficiently high Reynolds numbers, we observed similar wake structures (unsteady shedding of double-sided hairpin shaped vortices) for all high yaw angles. A new wake structure was discovered for $\theta = 60^\circ$ and $Re \geq 175$. It is characterized by the periodic shedding of double-sided hairpin shaped vortices along a straight horizontal line. Although the comparison between the freely falling cylinder and the fixed inclined cylinder was limited to a few configurations, a good agreement was generally observed. The most surprising feature was the good agreement between the torque oscillation frequency in the fixed case and the yaw angle oscillation frequency during fluttering motion in the freely falling case.

We plotted the hydrodynamic force and torque as a function of Re and θ and attempted to fit these plots with force and torque correlations. This is an important result of this study as these correlations are meant to be later used in higher scale models as, e.g., Euler/Lagrange and Euler/Euler models, for the simulation of dilute suspensions of cylinders. The correlation given by Rosendahl [49] based on the IP is shown to be poorly accurate to describe the drag force exerted on a yawed cylinder. For the aspect ratio $L/D = 3$ studied and $25 \leq Re \leq 100$, a law derived in the Stokes regime seems to be better suited. We proposed simple laws derived in the Stokes regime that do not explicitly require the drag coefficient when the cylinder is aligned with the flow direction. These laws give reasonable agreement with numerical results for both the drag force and the lift force. The IP is also shown to be

poorly accurate to describe the drag force on a yawed cylinder for $100 \leq Re \leq 200$. Instead, we derived an empirical relationship that gives better agreement with our numerical results. Based on a symmetry assumption, we also derived a law for the torque that explicitly involves the normal force. For all Re examined, our torque law matches well our numerical results as long as we can predict properly the position of the hydrodynamic centre. We showed that this can be done using a fit that involves trigonometric functions. Based on the results of Cox [13] and our numerical results, we suggested 2 different laws in the ranges $Re \in [25, 150]$ and $Re \in [150, 250]$, respectively. The resulting predictions of these 2 laws give full satisfaction.

The range of validity of the IP as a function of L/D needs further investigation. Indeed for very high Re and L/D , the IP supplies accurate predictions [23, 66]. We can hence redo a similar study for higher L/D if we wish to further challenge the IP. Therefore the flow map shown in FIG. 3 would be enriched with a third parameter, the aspect ratio L/D . This would also give us the opportunity to compare our numerical results to the results of Ramberg [47]. Another interesting perspective is the physical explanation of the agreement between the Stokes based laws and the numerical results even for moderate Reynolds number. Distinguishing viscous and pressure contributions in the loads will be a key factor in better explaining the observed agreement. The specific configuration of the cylinder aligned with the flow also deserves further investigation, first to provide a lower bound for the drag coefficient, second to investigate the impact of L/D on boundary layer detachment and third to study the bifurcation scenario. For instance, Ern et al. [16] gave a threshold value for the Reynolds number Re_c associated with the first bifurcation for a disk $L/D \leq 1$ aligned with the flow. It is estimated as $Re_c = 116.5(1 + L/D)$.

Finally, we intend to investigate the problem of a freely moving 3D cylinder as, e.g., a heavy cylinder settling in a quiescent light fluid. We already performed such a study for angular polyhedra in Rahmani and Wachs [46]. Our numerical method is actually better suited to moving particles than to flows past fixed obstacles (see [17, 46, 59] among other references), although it can be used (as here) for the flow past a single or multiple obstacles [15]. While improvements to our numerical method are always suitable (higher order schemes, enhanced parallel performance), particle-resolved simulations are computationally demanding. The finest mesh selected in this work comprised 100 millions of grid cells. The corresponding computation ran for several days/weeks on 512 cores. However, as shown in this paper, inertial regimes lead to thin boundary layers around freely moving or fixed

particles that need to be captured properly. This puts some stress on the mesh size, which consequently increases the number of cells, in particular in a constant grid size approach as ours. An alternative would be to re-develop our Fictitious Domain method in an octree/adaptive mesh refinement [44] framework that would give us the ability to refine the mesh wherever required and to loose it away from solid particles and/or from regions of the flow with strong velocity gradients.

Acknowledgements This work was granted access to the HPC resources of GENCI under the allocation c20162a7728 and c20172a7728. JLP wish to thank Jacques Magnaudet for fruitful discussions regarding this subject.

Appendix A: Technical aspects of the numerical procedure

1. Solution algorithm

The three-dimensional unsteady incompressible Navier-Stokes equations with appropriate initial conditions and boundary conditions subject to the rigid body motion (motionless in the particular case of the flow problem examined in this paper) constraint in the region occupied by the cylinder denoted P (and filled with fictitious fluid) are solved by a first-order Marchuk-Yanenko time splitting solution algorithm. We use a simple two-step splitting: at each discrete time, we (i) solve the incompressible Navier-Stokes equations and then (ii) solve the fictitious domain problem to enforce the rigid body motion constraint. Problem (i) is solved with a classical L^2 -projection scheme while Problem (ii) is a saddle-point problem solved by a Uzawa/conjugate gradient algorithm [22, 61]. While the solution of Problem (i) is second-order accurate in time, the whole solution algorithm is first-order accurate in time only due to the first-order Marchuk-Yanenko splitting. Please note that the presence of the explicit forcing term $\pm \mathbf{f}^n$ on the right-hand side of (A1a) and (A2a) significantly enhances the coupling of the two sub-problems [61]. \mathbf{f} is simply a smoothed equivalent of the distributed Lagrange multiplier $\boldsymbol{\lambda}^n$ associated to the constraint $\mathbf{u} = \mathbf{0}$ in P [61]. At each discrete time t^{n+1} where n refers to the discrete time index, we know the fluid velocity \mathbf{u}^n , the pressure p^n and the distributed Lagrange multiplier $\boldsymbol{\lambda}^n$ in P , and solve the following equations for \mathbf{u}^{n+1} , p^{n+1} and $\boldsymbol{\lambda}^{n+1}$ as follows:

- A classical L^2 -projection scheme for the solution of the Navier- Stokes problem: find

$\mathbf{u}^{n+1/2}$ and p^{n+1} such that

$$\rho \frac{\tilde{\mathbf{u}}^{n+1/2} - \mathbf{u}^n}{\Delta t} - \frac{\mu}{2} \nabla^2 \tilde{\mathbf{u}}^{n+1/2} = -\nabla p^n + \frac{\mu}{2} \nabla^2 \mathbf{u}^n - \frac{\rho}{2} (3\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^{n-1}) - \mathbf{f}^n, \quad (\text{A1a})$$

$$\nabla^2 \psi^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \tilde{\mathbf{u}}^{n+1/2}, \quad (\text{A1b})$$

$$\mathbf{u}^{n+1/2} = \tilde{\mathbf{u}}^{n+1/2} - \frac{\Delta t}{\rho} \nabla \psi^{n+1}, \quad (\text{A1c})$$

$$p^{n+1} = p^n + \psi^{n+1} - \frac{1}{2} \frac{\mu \Delta t}{\rho} \nabla^2 \psi^{n+1} \quad (\text{A1d})$$

where ρ denotes the fluid density, μ the viscosity and ψ^{n+1} the auxiliary potential. In (A1a), the advection and viscous terms are discretized in time by an Adams-Bashford scheme and a Crank-Nicholson scheme, respectively.

- A fictitious domain problem: find \mathbf{u}^{n+1} and $\boldsymbol{\lambda}^{n+1}$ such that

$$\rho \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}}{\Delta t} + \boldsymbol{\lambda}^{n+1} = \mathbf{f}^n, \quad (\text{A2a})$$

$$\mathbf{u}^{n+1} = \mathbf{0} \quad \text{in } P \quad (\text{A2b})$$

It can easily be shown [46] that the hydrodynamic force \mathbf{F} and torque \mathbf{T} exerted on the cylinder can be expressed as volume integrals over the cylinder P of the distributed Lagrange multiplier $\boldsymbol{\lambda}$ as follows:

$$\mathbf{F} = \int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_P \boldsymbol{\lambda} dV \quad (\text{A3a})$$

$$\mathbf{T} = \int_S \mathbf{r} \times \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_P \mathbf{r} \times \boldsymbol{\lambda} dV \quad (\text{A3b})$$

where $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e}$ is the Newtonian stress tensor, \mathbf{I} the identity tensor, $\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ the strain-rate tensor, $\nabla\mathbf{u}$ the velocity gradient tensor, \mathbf{r} the position vector relative to the solid mass centre and \mathbf{n} the unit vector normal to the cylinder surface S .

2. Space discretization scheme

(A1a)-(A1d) are solved on a cartesian grid with a Finite Volume approach. The pressure is cell-centered while the velocity component i is located at the cell face perpendicular to direction i . A second order central discretization scheme is employed for the diffusion term

while the convective term is treated with a total variation diminishing (TVD) scheme and a Superbee flux limiter. Without any immersed solid body in the flow domain, the overall space accuracy of the fluid discretization scheme is 2nd order. However due to the presence of the immersed boundary solid bodies, the method is not fully second order in space [61].

In order to enforce the zero velocity constraint inside the cylinder, we use a collocation point method in which the basis functions for the distributed Lagrange multiplier are delta Dirac functions [22, 61]. A set of Lagrangian points is distributed along the surface and inside the cylinder. Interior points are distributed on the staggered grid at the location of velocity points. Distributing points uniformly along the cylinder surface, called boundary points, is much more challenging. The detailed method developed in the PeliGRIFF code is described in SM [SMd]. To summarize, we divide the cylinder in two main areas : its length and its two end disks. The surface area defined along the cylinder length is mapped using a diamond-shaped mesh while the two end disks are mapped with a specific spiral distribution. This methodology ensures that boundary points are close to uniformly distributed. Regardless of the body shape, the quasi uniform distribution of boundary points property has proven to be important for the accuracy of the computed solution [61]. Finally, a multi-dimensional 2nd order quadratic reconstruction operator is used to enforce the zero velocity constraint at the Lagrangian points located on the cylinder surface. The construction of the 3D stencil of this operator relies on the orientation of the outward normal vector to the particle boundary and is a key ingredient of the space discretization scheme. It has also proven to significantly improve the accuracy of the computed solution. The accuracy of our scheme with immersed solid bodies is between first and second order, depending on the flow configuration, but the main asset of the multi-dimensional 2nd order quadratic reconstruction operator is that it does not require any hydrodynamic radius calibration neither for spheres nor for any other shape [61], and can hence be safely applied to a cylinder here.

The explicit forcing term \mathbf{f}^n in (A1a) and (A2a) is computed using the multi-dimensional 2nd order quadratic reconstruction operator for boundary points and is smoothed out for interior points using a simple hat function defined on a support that spans three grid points in each direction. This simple procedure has proven to be efficient in all our previous studies (see Dorai et al. [15], Rahmani and Wachs [46], Wachs et al. [61] among others) and shares some similarities with the regularized Dirac delta function used by Uhlmann [56] and Kempe and Fröhlich [31], although in these works a regularized Dirac delta function is employed

in an IBM fashion to enforce the no-slip boundary condition on boundary points only, leaving interior points free to flow as a fluid. In our method, since the distributed Lagrange multipliers $\boldsymbol{\lambda}^{n+1}$ that implicitly enforces the zero velocity constraint in the cylinder use actual Dirac delta functions as basis functions, the a posteriori computation of the hydrodynamic force \mathbf{F} and torque \mathbf{T} exerted on the cylinder is rather straightforward. In fact, computing \mathbf{F} and \mathbf{T} at time t^{n+1} translates into summing the distributed Lagrange multipliers as:

$$\mathbf{F}^{n+1} = \int_P \boldsymbol{\lambda}^{n+1} dV = \sum_{j=0}^{N_i-1} \boldsymbol{\lambda}_j^{n+1} + \sum_{k=0}^{N_b-1} \boldsymbol{\lambda}_k^{n+1} \quad (\text{A4a})$$

$$\mathbf{T}^{n+1} = \int_P \mathbf{r} \times \boldsymbol{\lambda}^{n+1} dV = \sum_{j=0}^{N_i-1} \mathbf{r}_j \times \boldsymbol{\lambda}_j^{n+1} + \sum_{k=0}^{N_b-1} \mathbf{r}_k \times \boldsymbol{\lambda}_k^{n+1} \quad (\text{A4b})$$

where N_i and N_b denote the number of interior points and number of boundary points, respectively.

Appendix B: Details on the computation of the Strouhal numbers using Discrete Fourier Transform

In this appendix we give a detailed explanation on the way Strouhal numbers are computed. Once the temporal signal of the lift coefficient is obtained we perform a Discrete Fourier Transform (DFT) of the signal using the python scipy library. The DFT is performed from at least $t = 100$ (in order to avoid the temporary regime) and until at least $t = 200$. Moreover we also check that the signal reaches at least 10 periods. A long simulation time is a necessary requirement in order to obtain a good estimate of the dimensionless frequency. Indeed the dimensional spectral resolution is $1/(n\Delta t)$ where n is the number of simulated time steps used in the DFT and Δt the time step [8]. The error made when performing the DFT is estimated as $\pm 1/2 \times D/U \times 1/(n\Delta t)$.

FIG. 24 displays the time evolution of the lift coefficient along z for $L/D = 3$, $Re = 150$, $\theta = 90^\circ$. A single frequency appears in the wake which is confirmed by the DFT (figure 24). Indeed a single peak is visible in the frequency space. The Strouhal number is estimated as $St = 0.118 \pm 0.002$

When a single frequency appears in the wake its identification is straightforward as shown above. The identification of dominants frequencies becomes much more difficult when the signal is quasi-periodic or chaotic [8]. For the later case the Fourier spectrum becomes

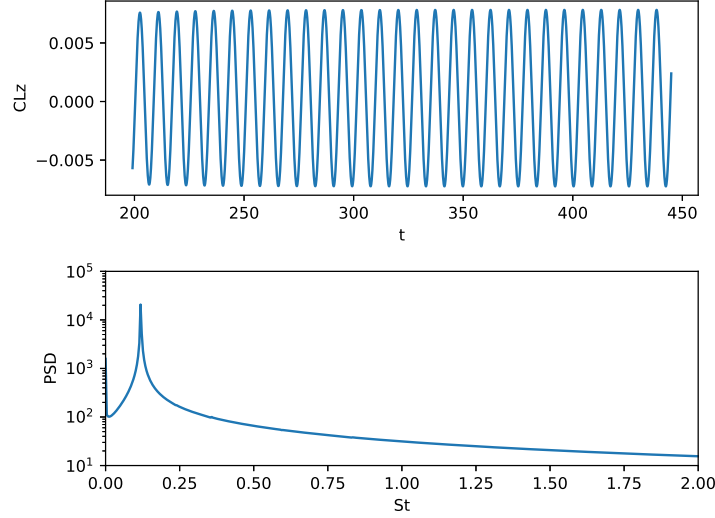


FIG. 24: $L/D = 3$, $Re = 150$, $\theta = 90^\circ$. Above : C_{Lz} as a function of time. Below : Power spectra density (PSD) of C_{Lz} .

continuous and it becomes very difficult to identify the dominant frequency. Fortunately, in the present study, the Reynolds number are moderate and the wake past yawed cylinder is not fully chaotic. In most cases it is possible to identify dominant frequency as detailed here.

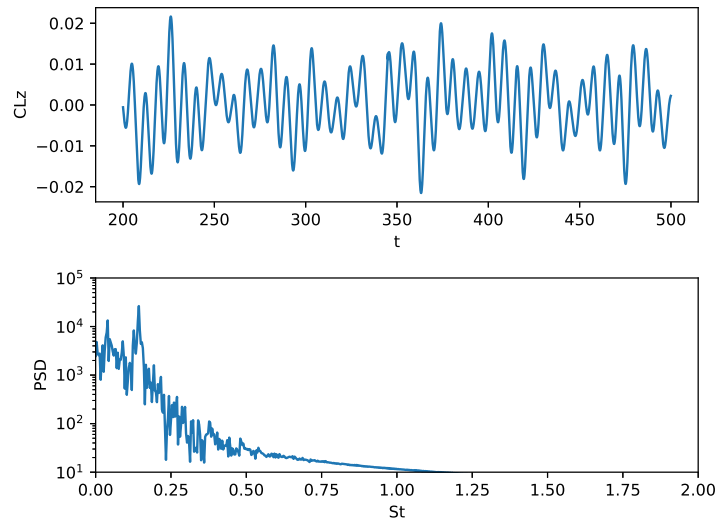


FIG. 25: $L/D = 3$, $Re = 250$, $\theta = 75^\circ$. Above : C_{Lz} as a function of time. Below : Power spectra density (PSD) of C_{Lz} .

Figure 25 displays the lift force oscillations along z for $L/D = 3$, $Re = 250$, $\theta = 75^\circ$.

Compared to figure 24, the amplitudes of the oscillations are not constant. However a dominant frequency is visible in the Fourier space with a peak at $St = 0.143 \pm 0.002$. This frequency is not the only which appears : a smaller one also is visible ($St = 0.040 \pm 0.002$) and one can note that the spectrum is almost continuous. This feature means that many frequency appears in the wake and the flow is thus quasi-steady or close to be chaotic [8].

Appendix C: Flow past a $L/D = 3$ cylinder : details on some regimes

1. Cylinder oriented perpendicularly to the flow direction

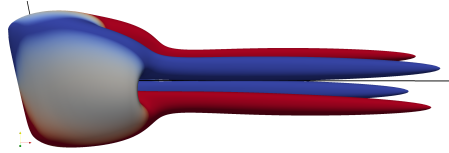


FIG. 26: Two steady symmetric counter-rotating vortex pairs at ($\theta = 90^\circ$, $Re = 100$). $Q = 0.01$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.1 to 0.1 . The dark lines starting at the center of the cylinder indicate the cylinder axis and the streamwise direction.

Two steady symmetric counter-rotating vortex pairs Up to $Re \approx 120$ (FIG. 3, \square), the flow structure past the solid body oriented perpendicularly to the flow direction presents two steady symmetric counter-rotating vortex pairs displayed in FIG. 26. The regime is steady and symmetric with respect to the (x, y) vertical plane and (x, z) horizontal plane. The singular symmetric nature of the four vortices induces a zero-value of the lift force in both y and z directions.

Periodic shedding of 2 counter-rotating vortex pairs From $Re \gtrsim 125$, the flow becomes unsteady and a periodic symmetric vortex shedding manifests, as presented in FIG. 27. The nature of the symmetry described above for ($\theta = 90^\circ$, $Re < 120$) is altered and the (x, y) vertical symmetry is broken by an unsteady mode while the horizontal symmetry plane is kept. FIG. 29a displays the force diagram for $Re = 125$. The lift force along z oscillates around a near-zero mean value ($\overline{C_{Lz}} = 1.136 \times 10^{-4}$) with a low amplitude. The unsteady character of the solution has a weak impact on the drag force and the attractor looks like a vertical line. The lift force along y is zero in agreement with the wake symmetry. The

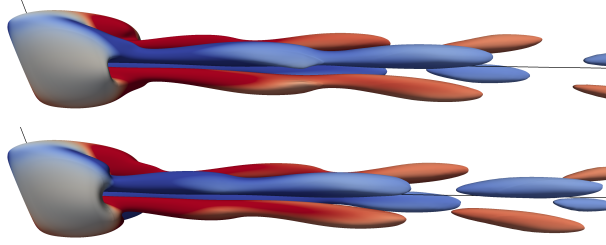


FIG. 27: Periodic shedding of two symmetric counter-rotating pairs at ($\theta = 90^\circ$, $Re = 125$). The time interval between two frames is $\Delta t = 5$. $Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

range of existence of this mode as a function of Re is narrow (FIG. 3, \odot). Moreover one may note that the amplitude of the oscillation of C_{Lz} in that regime keeps increasing even after a time 500. Those two remarks in mind, this regime can be interpreted as a transitional one between a steady regime and the periodic regime described hereafter.

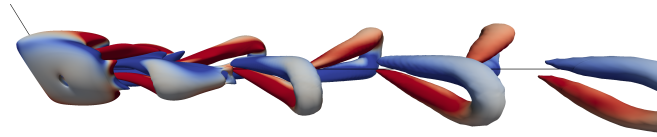


FIG. 28: Periodic shedding of double-sided horizontal hairpin vortices at ($\theta = 90^\circ$, $Re = 150$). $Q = 0.008$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

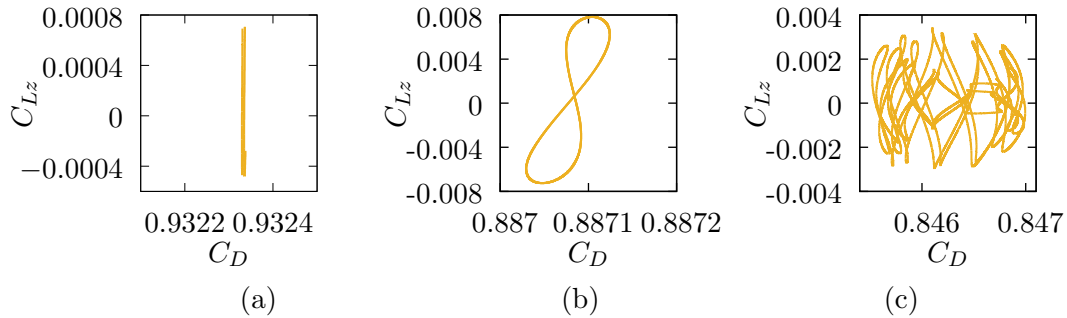


FIG. 29: $C_D - C_{Lz}$ diagrams for the symmetric shedding of double-sided hairpin vortices at ($\theta = 90^\circ$): (a) $Re = 125$, (b) $Re = 150$, (c) $Re = 175$.

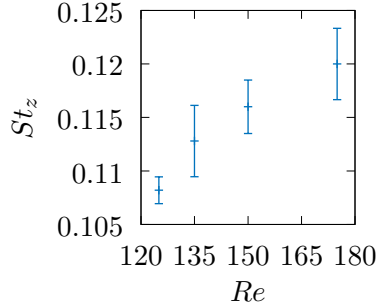


FIG. 30: Strouhal number as a function of Re . The meaning of the error bars can be found in *Section 2.2.1*.

Periodic shedding of symmetric double-sided hairpin shaped vortices A transition occurs for $Re \approx 135$. The wake pattern exhibits hairpin shaped vortex shedding (FIG. 3, ●). FIG. 28 shows the flow structure obtained for $Re = 150$. The hairpins are double-sided in the sense that alternated vortices of opposite rotational direction but of similar intensity are shed periodically.

The $C_D - C_{Lz}$ diagram for $Re = 150$ presented in FIG. 29b shows an attractor displaying an "eight" shape. The lift force mean value in the z direction is $\overline{C_{Lz}} = 2.5 \cdot 10^{-4}$ and remains small. This implies, as observed on the wake pattern, that the hairpin vortices shed from both sides of the cylinder have approximatively the same intensity. The C_{Lz} oscillations amplitude is large in comparison to the C_D oscillations amplitude. Moreover, the C_{Lz} oscillations amplitude is one order of magnitude larger than the C_{Lz} oscillations amplitude observed for ($\theta = 90^\circ, Re = 125$). The emergence of hairpin vortices is related to an increase of the vortex intensity and is a distinguishable signature of the unsteady mode. The C_D and C_{Lz} oscillations are slightly out of phase since the extrema of C_D occurs slightly shifted with respect to the extrema of C_{Lz} . C_D oscillates with a dimensionless frequency equal to twice the leading dimensionless frequency in agreement with the symmetry observed for the wake pattern. The amplitude of oscillations of C_D is negligible in comparison with the C_D mean value. The lift force in the y direction is almost negligible in that case.

The Strouhal number associated with the oscillation of the lift in this regime is a slowly increasing function of the Reynolds number (FIG. 30). Its value stays close to $St_z \approx 0.12$ and is comparable to the one encountered for larger aspect ratio [28] and in the wake of other bluff body such as spheres [30].

The $Re = 175$ force diagram shown in FIG. 29c reveals the appearance of several fre-

quencies in the wake. Compared to $Re = 150$ a low frequency $St_{z,\text{low}} \approx 0.039$ appears which modulates the higher frequency. The drag force DFT shows a dominant dimensionless frequency $St_x \approx 0.27$ close to twice the leading dimensionless frequency in the transverse z direction and a low dimensionless frequency $St_{x,\text{low}} \approx 0.026$. This low frequency has a strong effect on the amplitude of drag oscillations since it increases of approximately one order of magnitude with respect to $Re = 150$. This suggest that the magnitude of the hairpin shaped vortices are oscillating with a low frequency. The y lift force amplitude, i.e., C_{Ly} , is still weak (not shown here for the sake of conciseness) and of the order of one hundred times lower than C_{Lz} but is nonetheless increasing with Re . The regime observed for $Re = 175$ can be seen as a transitional regime separating the shedding of symmetric hairpin vortices regime and the fully 3D wake regime.

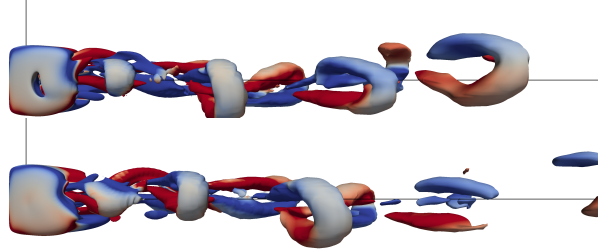


FIG. 31: 3D unsteady regime dominated by the shedding of horizontal double-sided hairpin vortices at $(\theta = 90^\circ, Re = 250)$. The time interval between two frames is $\Delta t = 5$. $Q = 0.008$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

Unsteady shedding of asymmetric double-sided hairpin shaped vortices For $Re \gtrsim 200$, the force and torque diagrams reveal chaotic solutions and a clear loss of the (x, z) reflectional planar symmetry. This wake is characterized by pseudo-chaotic flow structures (FIG. 3, *) and asymmetric double-sided hairpin vortices displayed in FIG. 31. Even if the solution becomes chaotic, the non-isotropic body geometry does not create a fully 3D behaviour of the flow in the near-wake. However the asymmetry is more pronounced as Re increases. Finally the wake patterns for $\theta = 90^\circ$ is similar to the wake pattern observed for lower θ in the range $[70^\circ, 90^\circ]$.

For $Re = 250$. the C_D and C_{Lz} oscillations amplitudes are at least one order of magnitude larger than the ones obtained for $(\theta = 90^\circ, Re = 175)$. The vortex shedding process is much

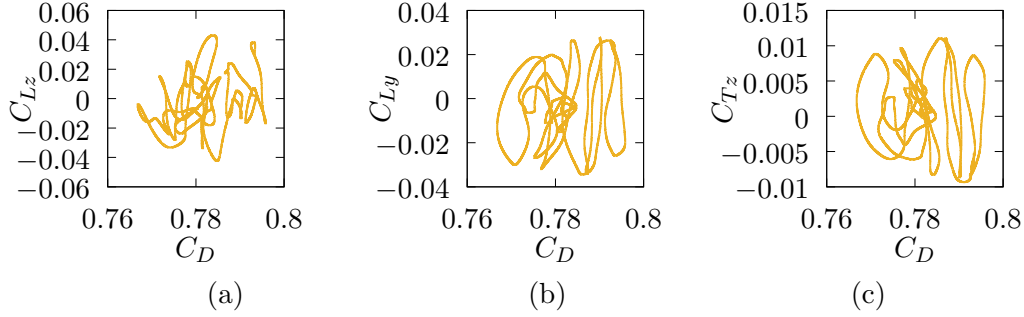


FIG. 32: Force and torque coefficient diagrams for the 3D unsteady regime dominated by the shedding of horizontal double-sided hairpin vortices at $(\theta = 90^\circ, Re = 250)$.

more energetic. The y lift force oscillation amplitude increases with Re to reach a value close to the z lift force oscillation amplitude, in other words C_{Ly} is getting closer to C_{Lz} . For $Re = 250$, the more energetic unsteady mode has a dimensionless frequency $St_z \approx 0.15$, approximately 32% higher than for all lower Re unsteady regimes investigated. However, due to the numerous other frequencies appearing in the wake, the definition of St is not straightforward. In the x direction the high frequency oscillations observed for $Re = 150$ have almost disappeared and the dominant mode is the low frequency mode observed for $Re = 175$. The y lift force DFT gives $St_{y,Re=200} \approx 0.04$ and $St_{y,Re=250} \approx 0.05$. These pseudo-dominant dimensionless frequencies are at least twice smaller than the pseudo-dominant dimensionless frequency in the z direction. For $Re \leq 175$, the associated torque magnitude in the z direction remains weak (not shown here for the sake of conciseness) while for $Re \gtrsim 200$ the associated diagram presented in FIG. 32c shows a distinct and large torque. The torque oscillations amplitude, i.e., the C_{Tz} oscillations amplitude, increases with Re and reaches a value close to the C_{Ly} oscillations amplitude at $Re = 250$. Furthermore the C_{Tz} frequency is the same as C_{Ly} .

2. Onset of unsteadiness for $60^\circ \leq \theta < 70^\circ$

Steady shedding of a pair of counter-rotating vortices The single pair of counter-rotating vortices regime is also the regime that prevails for $Re \lesssim 125$ and lower yaw angles $0^\circ < \theta \leq 60^\circ$. In fact, FIG. 33a shows a pair of counter-rotating vortices in the wake of the cylinder for $(\theta = 45^\circ, Re = 75)$ that looks like the arms of a squid. This regime has been observed for the flow past a sphere by Johnson and Patel [30], past $L/D \geq 20$ long cylinders with flat ends

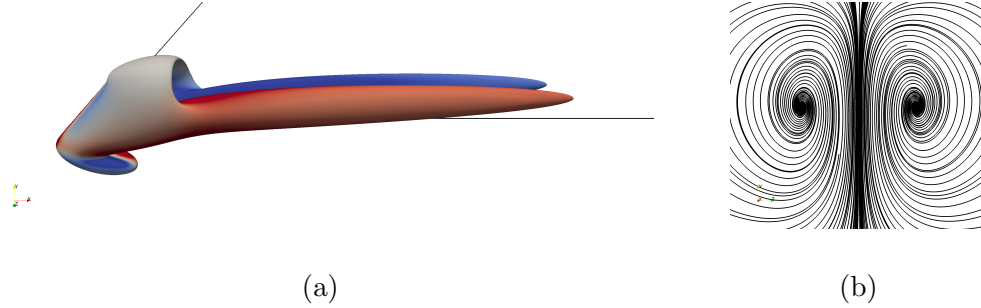


FIG. 33: One steady counter-rotating vortex pair at $(\theta = 45^\circ, Re = 75)$: (a) $Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction. (b) Streamlines in a (y, z) plane perpendicular to the x streamwise direction located at $5D$ from the x coordinate of the cylinder mass centre.

by Ramberg [47] and past long pointed-end cylinders at high Reynolds numbers and small yaw angles by [19]. This pattern is also called bifid wake by [21] in the case of the flow past a sphere. In FIG. 33a, the region below the cylinder also presents a distinctive bulge of opposite vorticity contours. The entire wake keeps a (x, y) reflectional planar symmetry. FIG. 33b shows the streamlines of the projected velocity in a plane perpendicular to the streamwise direction located a few diameters downstream of the cylinder for $(\theta = 45^\circ, Re = 75)$. We can clearly see two opposite longitudinal vortices.

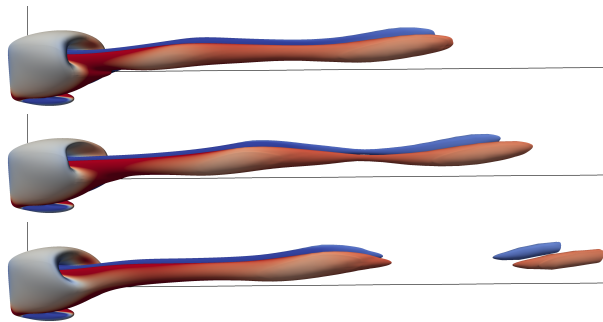


FIG. 34: Periodic shedding of one counter-rotating vortex pair at $(\theta = 65^\circ, Re = 135)$. The time interval between two frames is $\Delta t = 5$. $Q = 0.004$ isosurfaces are coloured by the longitudinal vorticity ranging from -0.2 to 0.2 . The dark lines starting at the centre of the cylinder indicate the cylinder axis and the streamwise direction.

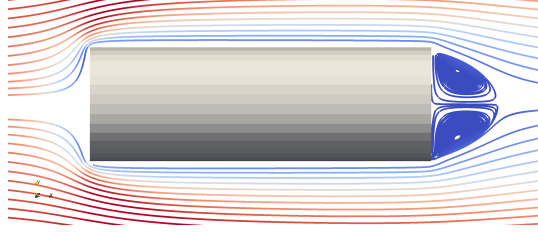


FIG. 35: Steady toroidal vortex downstream of a $\theta = 0^\circ$ cylinder at $Re = 50$. Streamlines pattern (x, y) plane perpendicular to the z transverse containing the cylinder axis are coloured by the dimensionless axial velocity (dark blue $u_x = 0$, red $u_x = 1$).

Periodic shedding of a pair of counter-rotating vortices This regime occurs for ($\theta = 65^\circ$; $Re = 135$) and is characterized by a vertical symmetry plane. FIG. 34 illustrates the wake pattern observed in that case. In comparison to the steady shedding of one counter-rotating vortex pair, the pair of vortices here twists with respect to their centre line. The dimensionless vortex shedding frequency is $St_y \approx 0.067$.

3. Cylinder oriented parallel to the flow direction

Standing toroidal eddy The aligned cylinder case completes the flow map shown in FIG. 3. Due to the specific symmetry of this configuration, the wake stays axisymmetric for all considered Re . The wake is characterized by a steady recirculating eddy as illustrated in FIG. 35 for ($\theta = 0^\circ$, $Re = 50$). This is a distinguishable feature of the flow past a bluff body at Reynolds numbers higher than 10 [7, p. 259]. When the body is angular with salient edges, the detachment of the boundary layer usually occurs at the edge of the body [7, p. 329].

The length of the recirculation region l_r increases with Re as plotted in FIG. 36. We also plot in FIG. 36 the recirculation length as a function of Re for an aspect ratio $L/D = 1$. For both aspect ratios, l_r increases as \sqrt{Re} . In the case of the flow past a sphere, l_r versus Re follows a logarithmic law [54], while a linear law has been found in the case of the flow past an infinite cylinder [12]. The mathematical form of the relationship between l_r and Re thus depends on the geometry of the obstacle.

The transition from the steady recirculating eddy regime to a periodic or unsteady vortex shedding regime for the flow past a $L/D = 3$ cylinder is expected to occur for higher Reynolds

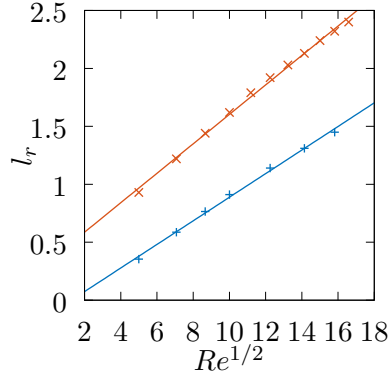


FIG. 36: Recirculation length l_r of an aligned cylinder with the flow direction with respect to the square root of the Reynolds number. + : $L/D = 3$ cylinder. \times : $L/D = 1$ cylinder (see SM [SMf]). The solid lines correspond to fits of our numerical results of the $aRe^{1/2} + b$ form, where a and b are the fitting parameters.

number as observed for the flow past a $L/D = 1$ cylinder (see SM [SMf]).

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[SMf] See Supplemental Material attached for a description of the boundary-fitted results.

[SMd] See Supplemental Material attached for a description of the distribution of lagrangian points at the surface of a cylinder.

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- [Note1] Note1. For instance if there are two modes, one with a low frequency and one with a high frequency we write St_{low} and St respectively in order to differentiate them.
- [Note2] Note2. The increase of numerical errors for low Reynolds number flow past immersed boundaries is also observed by Kempe and Fröhlich [31] and [?]. In their cases those are a direct consequence of IBM forcing before the implicit step of the Crank-Nicholson method. This creates an error on the forcing term that scales as $\mathcal{O}(\Delta t \mu / \rho)$.
- [Note3] Note3. This bifurcation might already exist since the mean of the lift force observed in

the periodic regimes is not exactly zero (*Appendix C 1*). A more detailed study could clarify if the non-zero mean value is due to a real bifurcation or to numerical errors.

- [Note4] Note4. Single pairs of steady counter-rotating vortices are found for ($Re = 25, \theta = 75^\circ$) and ($Re = 125, \theta = 70^\circ$). This regime is presented in the next section and detailed in *Appendix C 2*.
- [Note5] Note5. We have purposely considered a range of yaw angles larger than the one discussed in the present section, since the added regimes ($\theta = 65^\circ$) have slightly different wake patterns but comparable Strouhal numbers in the range $135 \leq Re \leq 160$.
- [Note6] Note6. It is possible to include higher order terms, but it does not improve significantly the model. Indeed one of the drawbacks of slender body theory is that, using an expansion in $1/\log(2L/D)$, the expansion is not very accurate for small L/D and rapidly diverges as L/D goes to 0.5 Batchelor [6].
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